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The investigations summarized in this report deal with: (a) adaptive dual control of systems with unknown parameters; (b) estimation and control of hybrid stochastic systems; (c) distributed estimation in systems with measurements of uncertain origin; and (d) solution of continuous-time hybrid stochastic differential equations.

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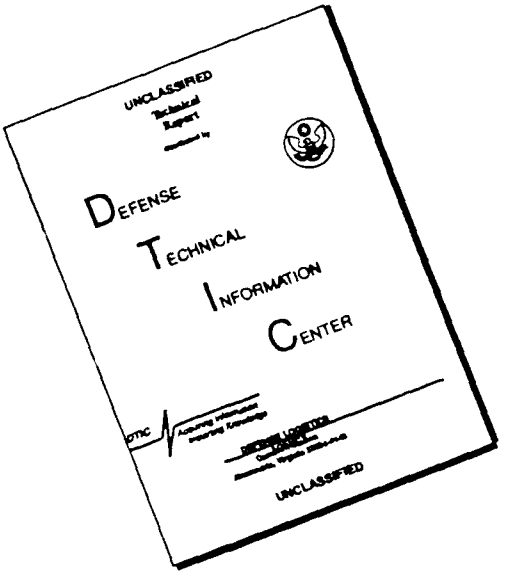


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Annual Report

Grant AFOSR-88-0202

Stochastic Adaptive Control

and

Estimation Enhancement

Y. Bar-Shalom

September 1989

The main results obtained and published during the period covered by this report, August 1988 - July 1989, are described below together with references given to the corresponding publication.

1. The Interacting Multiple Model Algorithm for Systems with Markovian Switching Coefficients, (Henk A. Blom and Yaakov Bar-Shalom, IEEE Transactions on Automatic Control Vol. 33, No. 8, August 1988)

An important problem in filtering for linear systems with Markovian switching coefficients (dynamic multiple model systems) is the one of management of hypotheses, which is necessary to limit the computational requirements. A novel approach to hypotheses merging has been developed for this problem. The novelty lies in the timing of hypotheses merging. When applied to the problem of filtering for a linear system with Markovian coefficients this yields an elegant way to derive the interacting multiple model (IMM) algorithm. Evaluation of the IMM algorithm makes it clear that it performs very well at a relatively low computational load. These results imply a significant change in the state of the art of approximate Bayesian filtering for systems with Markovian coefficients.

2. Failure Detection Via Recursive Estimation for a Class of Semi-Markov Switching Systems, (L. Campo, P. Mookerjee and Y. Bar-Shalom, Proceedings 1988 IEEE CDC, Austin, Texas)

An area of current interest is the estimation of the state of discrete-time stochastic systems with parameters which may switch among a finite set of values. The parameter switching process of interest is modeled by a class of semi-Markov chains. This class of processes is useful in that it pertains to many areas of interests such as the failure detection problem, the target tracking problem, socio-economic problems and in the problem of approximating nonlinear systems by a set of linearized models. It is shown in this paper how the transition probabilities, which govern the model switching at each time step, can be inferred via the evaluation of the conditional distribution of the sojourn time. Following this, a recursive state estimation algorithm for dynamic systems with noisy observations and changing structures, which uses the conditional sojourn time distribution, is derived and and applied to a failure detection problem.

3. Distributed Adaptive Estimation with Probabilistic Data Association, (K.C. Chang and Y. Bar-Shalom, Automatica, Vol. 25, No. 3, pp. 359-369, 1989)

   The probabilistic data association filter (PDAF) estimates the state of a target in a cluttered environment. This suboptimal Bayesian approach assumes that the exact target and measurement models are known. However, in most practical applications, there are difficulties in obtaining an exact mathematical model of the physical process. In this paper, the problem of estimating target states with uncertain measurement origins and uncertain system models in a distributed manner is considered. First, a scheme is described for local processing, then the fusion algorithm which combines the local processed results into a global one is derived. The algorithm can be applied for tracking a maneuvering target in a cluttered and low detection environment with a distributed sensor network.

4. An Adaptive Dual Controller for a MIMO-ARMA System, (P. Mookerjee and Y. Bar-Shalom, IEEE Transactions on Automatic Control, Vol. 34, No. 7, July 1989)

   An explicit adaptive dual controller has been derived for a multiinput multioutput ARMA system. The plant has constant but unknown parameters. The cautious controller with a one-step horizon and a new dual controller with a two-step horizon are examined. In many instances, the myopic cautious controller is seen to turn off and converges very slowly. The dual controller modifies the cautious control design by numerator and denominator correction terms which depend upon the sensitivity functions of the expected future cost and avoids the turn-off and slow convergence. Monte-Carlo comparisons based on parametric and nonparametric statistical analysis indicate the superiority of the dual controller over the cautious controller.

5. Time-Reversion of a Hybrid State Stochastic Difference System, (Henk A.P. Blom and Yaakov Bar-Shalom, Proc. 1989 IEEE Intn'l. Conf. on Control & Applications, Jerusalem, Israel, April 1989 to appear in IEEE Trans. Info. Theory, 1990)

   This paper develops the reversion in time of a stochastic difference equation in a hybrid space, with a Markovian solution. The reversion is obtained by a martingale approach, which previously led to reverse time forms for stochastic equations with Gauss-Markov or diffusion solutions. The reverse time equations follow from a particular non-canonical martingale decomposition, while the reverse time equations for Gauss-Markov and diffusion solutions followed from the canonical martingale decomposition. The need for the non-canonical decomposition stems from the hybrid state space situation. The non-Gaussian discrete time situation leads to reverse time equations that incorporate a Bayesian estimation step.

6. A New Controller for Discrete-Time Stochastic Systems with Markovian Jump Parameters, (L. Campo and Y. Bar-Shalom, 11th IFAC World Congress, Tallinn, USSR, Aug. 1990

   A realistic stochastic control problem for hybrid systems with Markovian jump parameters may have the switching parameters in both the state and measurement equations. Furthermore, both the system state and the jump states may not be perfectly observed. Prior to this work the only existing implementable controller for this problem was based upon a heuristic multiple model partitioning (MMP) and hypothesis pruning. In this paper a stochastic control algorithm for stochastic systems with Markovian jump parameters was developed. The control algorithm is derived through the use of stochastic dynamic progamming and is designed to be used for realistic stochastic control problems, i.e., with noisy state obeservations. The state estimation and model identification is done via the recently developed Interacting Multiple Model algorithm. Simulation results show that a substantial reduction in cost can be obtained by this new control algorithm over the MMP scheme.

7. From Piecewise Deterministic To Piecewise Diffusion Markov Processes, (Henk A.P. Blom, Proc. IEEE CDC 1988)

Piecewise Deterministic (PD) Markov processes form a remarkable class of hybrid state processes because, in contrast to most other hybrid state processes, they include a jump reflecting boundary and exclude diffusion. As such, they cover a wide variety of impulsively or singularly controlled non-diffusion processes. Because PD processes are defined in a pathwise way, they provide a framework to study the control of non-diffusion processes along same lines as that of diffusions. An important generalization is to include diffusion in PD processes, but, as pointed out by Davis, combining diffusion with a jump reflecting boundary seems not possible within the present definition of PD processes. This paper presents PD processes as pathwise unique solutions of an Itô stochastic differential equation (SDE), driven by a Poisson random measure. Since such an SDE permits the inclusion of diffusion, this approach leads to a large variety of piecewise diffusion Markov processes, represented by pathwise unique SDE solutions.

# The Interacting Multiple Model Algorithm for Systems with Markovian Switching Coefficients

Henk A. P. Blom
Yaakov Bar-Shalom

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# The Interacting Multiple Model Algorithm for Systems with Markovian Switching Coefficients

HENK A. P. BLOM AND YAAKOV BAR-SHALOM

*Abstract*—**An important problem in filtering for linear systems with Markovian switching coefficients (dynamic multiple model systems) is the one of management of hypotheses, which is necessary to limit the computational requirements. A novel approach to hypotheses merging is presented for this problem. The novelty lies in the timing of hypotheses merging. When applied to the problem of filtering for a linear system with Markovian coefficients this yields an elegant way to derive the interacting multiple model (IMM) algorithm. Evaluation of the IMM algorithm makes it clear that it performs very well at a relatively low computational load. These results imply a significant change in the state of the art of approximate Bayesian filtering for systems with Markovian coefficients.**

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## I. INTRODUCTION

In this contribution we present a novel approach to the problem of filtering for a linear system with Markovian coefficients

$$x_t = a(\theta_t)x_{t-1} + b(\theta_t)w_t \tag{1}$$

with observations

$$y_t = h(\theta_t)x_t + g(\theta_t)v_t \tag{2}$$

$\theta_t$ is a finite state Markov chain taking values in $\{1, \cdots, N\}$ according to a transition probability matrix $H$, and $w_t$, $v_t$ are mutually independent white Gaussian processes. The exact filter consists of a growing number of linear Gaussian hypotheses, with the growth being exponential with the time. Obviously, for filtering we need recursive algorithms whose complexity does not grow with time. With this, the main problem is to avoid the exponential growth of the number of Gaussian hypotheses in an efficient way.

This hypotheses management problem is also known for several other filtering situations [10], [5], [6], [9], and [4]. All these problems have stimulated during the last two decades the development of a large variety of approximation methods. For our problem the majority of these are techniques that reduce the number of Gaussian hypotheses, by pruning and/or merging of hypotheses. Well-known examples of this approach are the detection estimation (DE) algorithms and the generalized pseudo Bayes (GPB) algorithms. For overviews and comparisons see [14], [7], [12], and [17]. None of the algorithms discussed appeared to have good performance at modest computational load. Because of that, other approaches have been also developed, mainly by way of approximating the model (1), (2). Examples are the modified multiple model (MM) algorithms [20], [7], the modified gain extended Kalman (MGEK) filter of Song and Speyer [13], [7], and residual based methods [19], [2]. These algorithms, however, also lack good performance at modest computational load in too many situations. In view of this unsatisfactory situation and the practical importance of better solutions, the filtering problem for the class of systems (1), (2) needed further study.

One item that has not received much attention in the past is the timing of hypotheses reduction. It is common practice to reduce the number of Gaussian hypotheses immediately after a measurement update. Indeed, on first sight there does not seem to be a better moment. However, in two recent publications [3], [1], this point has been exploited to develop, respectively, the so-called IMM (interacting multiple model) and AFMM (adaptive forgetting through multiple models) algorithms. The latter exploits pruning to reduce the number of hypotheses, while the IMM exploits merging. The IMM algorithm was the reason for a further evaluation of the timing of hypotheses reduction. A novel approach to hypotheses merging is presented for a dynamic MM situation, which leads to an elegant derivation of the IMM algorithm. Next Monte Carlo simulations are presented to judge the state of the art in MM filtering after the introduction of the IMM algorithm.

## II. TIMING OF HYPOTHESES REDUCTION

To show the possibilities of timing the hypothesis reduction, we start with a filter cycle from one measurement update up to and including the next measurement update. For this, we take a cycle of recursions for the evolution of the conditional probability measure of our hybrid state Markov process $(x_t, \theta_t)$. This cycle reads as follows:

$$P\{\theta_{t-1} | Y_{t-1}\} \xrightarrow{\text{Mixing}} P\{\theta_t | Y_{t-1}\} \tag{3}$$

if $P\{\theta_t | Y_{t-1}\} = 0$ prune hypothesis $\theta_t$,

$$p[x_{t-1}|\theta_{t-1}, Y_{t-1}] \xrightarrow{\text{Mixing}} p[x_{t-1}|\theta_t, Y_{t-1}] \tag{4}$$

$$p[x_{t-1}|\theta_t, Y_{t-1}] \xrightarrow{\text{Evolution}} p[x_t|\theta_t, Y_{t-1}] \tag{5}$$

$$P\{\theta_t|Y_{t-1}\} \xrightarrow{\text{Bayes}} P\{\theta_t|Y_t\} \tag{6}$$

$$p[x_t|\theta_t, Y_{t-1}] \xrightarrow{\text{Bayes}} p[x_t|\theta_t, Y_t]. \tag{7}$$

For output purposes, we can use the law of total probability

$$p[x_t|Y_t] = \sum_i p[x_t|\theta_t = i, Y_t] P\{\theta_t = i|Y_t\}. \tag{8}$$

Let us take a closer look at the derivation of the above cycle. As $v_t$ and $w_t$ are mutually independent, the Bayes formula, which represents (6) and (7), follows easily from (2). From the evolution of system (1) follows (5). The Chapman-Kolmogorov equation for the Markov *chain* $\theta_t$

$$P\{\theta_t = i|Y_{t-1}\} = \sum_j H_{ij} P\{\theta_{t-1} = j|Y_{t-1}\} \tag{9}$$

which represents (3), can be seen as a "mixing." To derive a representation of (4) we first introduce the following equation on the basis of the law of total probability:

$$p[x_{t-1}|\theta_t = i, Y_{t-1}] = \sum_j [p[x_{t-1}|\theta_{t-1} = j, \theta_t = i, Y_{t-1}] \cdot P\{\theta_{t-1} = j|\theta_t = i, Y_{t-1}\}]. \tag{10}$$

As $\theta_t$ is *independent* of $x_{t-1}$ *if* $\theta_{t-1}$ is known, we easily obtain

$$p[x_{t-1}|\theta_{t-1} = j, \theta_t = i, Y_{t-1}] = p[x_{t-1}|\theta_{t-1} = j, Y_{t-1}].$$

Substitution of this and of the following:

$$P\{\theta_{t-1} = j|\theta_t = i, Y_{t-1}\} = H_{ij} P\{\theta_{t-1} = j|Y_{t-1}\} / P\{\theta_t = i|Y_{t-1}\}$$

in (10) yields the desired representation of transition (4)

$$p[x_{t-1}|\theta_t = i, Y_{t-1}] = \sum_j H_{ij} P\{\theta_{t-1} = j|Y_{t-1}\} \cdot p[x_{t-1}|\theta_{t-1} = j, Y_{t-1}] / P\{\theta_t = i|Y_{t-1}\}. \tag{11}$$

Notice that the mixing of the densities in (11) is explicitly related to the above-mentioned Markov properties of $\theta_t$ and the conditional independence of $\theta_t$ and $x_{t-1}$, given $\theta_{t-1}$. According to the above filtering cycle there are at any moment in time $N$ densities on $R^n$ and $N$ scalars. The densities on $R^n$ are rarely Gaussian. Even if $p[x_0|Y_0]$ is Gaussian, then $p[x_t|\theta_t = i, Y_t]$ is in general a sum of $N^{t-1}$ weighted Gaussians (Gaussian mixture). Explicit recursions for these $N^t$ individual Gaussians and their weights can simply be obtained from the above filter cycle. Obviously, the $N$ times increase of the number of Gaussians during each filter cycle is caused by (4) only.

In the sequence of elementary transitions, (3) through (7), we can apply a hypotheses reduction either after (4), after (5), or after (7). We review these reduction timing possibilities for the fixed depth merging hypotheses reduction. This fixed depth merging approach implies that the Gaussian hypotheses, for which the Markov chain paths are equivalent during the recent past of some fixed depth, are merged to one moment-matched Gaussian hypothesis. The degrees of freedom in applying this fixed depth merging approach are the choice of the depth, $d$ ($\geq 1$), and the moment of application. If the application is immediately after each measurement update pass (7), it yields the GPB ($d + 1$) algorithms [14], [16]. In the next section we derive the IMM algorithm by applying the fixed depth merging approach with depth, $d = 1$, after each pass of (4). It can easily be verified that all other timing possibilities yield disguised versions of IMM and GPB algorithms. Merging after (5) with $d = 1$ yields a disguised but more complex IMM algorithm. Merging either after (4) or after (5) with $d \geq 2$ yields a disguised but more complex GPBd algorithm.

## III. The IMM Algorithm

The IMM algorithm cycle consists of the following four steps, of which the first three steps are illustrated in Fig. 1.

1) Starting with the $N$ weights $\hat{p}_i(t-1)$, the $N$ means $\hat{x}_i(t-1)$ and the $N$ associated covariances $\hat{R}_i(t-1)$, one computes the mixed initial condition for the filter matched to $\theta_t = i$, according to the following equations:

$$\bar{p}_i(t) = \sum_j H_{ij} \hat{p}_j(t-1), \text{ if } \bar{p}_i(t) = 0 \text{ prune hypothesis } i, \tag{12}$$

$$\hat{x}^i(t-1) = \sum_j H_{ij} \hat{p}_j(t-1) \hat{x}_j(t-1) / \bar{p}_i(t), \tag{13}$$

$$\hat{R}^i(t-1) = \sum_j H_{ij} \hat{p}_j(t-1) \{\hat{R}_j(t-1) + [\hat{x}_j(t-1) - \hat{x}^i(t-1)][\,.\,]^T\} / \bar{p}_i(t). \tag{14}$$

2) Each of the $N$ pairs $\hat{x}^i(t-1)$, $\hat{R}^i(t-1)$ is used as input to a Kalman filter matched to $\theta_t = i$. Time-extrapolation yields, $\bar{x}_i(t)$, $\bar{R}_i(t)$, and then, measurement updating yields, $\hat{x}_i(t)$, $\hat{R}_i(t)$.

3) The $N$ weights $\hat{p}_i(t)$ are updated from the innovations of the $N$ Kalman filters,

$$\hat{p}_i(t) = c \cdot \bar{p}_i(t) \cdot \|Q_i(t)\|^{-1/2} \exp\{-1/2\, \vartheta_i^T(t) Q_i^{-1}(t) \vartheta_i(t)\} \tag{15}$$

with $c$ denoting a normalizing constant

$$\vartheta_i(t) = y_t - h(i) \bar{x}_i(t) \tag{16}$$

$$Q_i(t) = h(i) \bar{R}_i(t) h^T(i) + g(i) g^T(i). \tag{17}$$

4) For output purpose only, $\hat{x}_t$ and $\hat{R}_t$ are computed according to

$$\hat{x}_t = \sum_i \hat{p}_i(t) \hat{x}_i(t) \tag{18}$$

$$\hat{R}_t = \sum_i \hat{p}_i(t) \{\hat{R}_i(t) + [\hat{x}_i(t) - \hat{x}_t][\,.\,]^T\}. \tag{19}$$

Only step 1) is typical for the IMM algorithm. Specifically, the mixing represented by (13) and (14) and by the interaction box in Fig. 1, cannot be found in the GPB algorithms. This is the key of the novel approach to the timing of fixed depth hypotheses merging that yields the IMM algorithm. We give a derivation of the key step 1).

Application of fixed depth merging with $d = 1$ implies that

$$p[x_{t-1}|\theta_{t-1} = i, Y_{t-1}] \sim N\{\hat{x}_i(t-1), \hat{R}_i(t-1)\}.$$

Substitution of this in (11) immediately yields (13) and (14), with

$$\hat{x}^i(t-1) \triangleq E\{x_{t-1}|\theta_t = i, Y_{t-1}\}$$

and

$$\hat{R}^i(t-1)$$

the associated covariance. Finally, we introduce the approximation,

$$p[x_{t-1}|\theta_t = i, Y_{t-1}] \sim N\{\hat{x}^i(t-1), \hat{R}^i(t-1)\}$$

which guarantees that all subsequent IMM steps fit correctly.

*Remark:* The IMM can be approximated by the GPB1 algorithm by replacing $\hat{x}_i(t-1)$ and $\hat{R}_i(t-1)$ in step 1) by $\hat{x}_{t-1}$ and $\hat{R}_{t-1}$. Together with (12) this approximates (13) and (14) in step 1) by, $\hat{x}^i(t-1) \simeq \hat{x}_{t-1}$ and $\hat{R}^i(t-1) \simeq \hat{R}_{t-1}$. These equations are equivalent to (13) and (14) if each component of $H$ equals $1/N$, which implies that $\theta_t$ is a sequence of mutually independent stochastic variables. The latter is hardly ever the case and we conclude that the reduction of the IMM to GPB1 leads to a significant performance degradation. Obviously, the computational loads of IMM and GPB1 are almost equivalent.

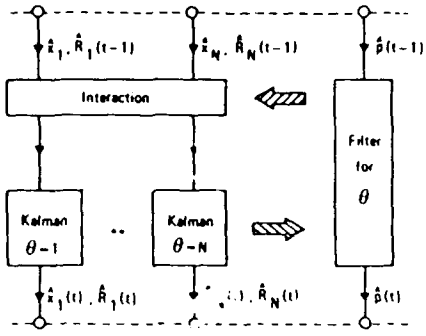


Fig. The IMM algorithm.

## IV. Performance of the IMM Algorithm

Presently a comparison of the different filtering algorithms for systems with Markovian coefficients with respect to their performance is hampered by the analytical complexity of the problem [16], [15]. Because of this, such comparisons necessarily rely on Monte Carlo simulations for specific examples. For our simulated examples we used the set of 19 cases that have been developed by Westwood [18]. To make the comparison more precise, we specify these cases and summarize the observed performance results. In all 19 cases both $x_t$ and $y_t$ are scalar processes, which satisfy $x_t = a(\theta_t)x_{t-1} + b(\theta_t)w_t + u(t)$ and $y_t = h(\theta_t)x_t + g(\theta_t)v_t$, with $\theta_t:\Omega \Rightarrow \{0, 1\}$, $u(t) = 10.\cos\{2\pi t/100\}$, $x_0$ a Gaussian variable with expectation 10 and variance 10, $P\{\theta_0 = 1\} = P\{\theta_0 = 0\} = 1/2$, while $H_{00} = (1 - 1/\tau_0)$ and $H_{11} = (1 - 1/\tau_1)$. The parameters $a$, $b$, $h$, $g$ and the average sojourn times $\tau_0$ and $\tau_1$ of these 19 cases are given in Table I.

The results of Westwood [18] show that, in all 19 cases the differences in performance of the GPB2 and the GPB3 algorithms are negligible, while in only seven cases (5, 6, 8, 16, 17, 18, 19) the differences in performance of the GPB1 and the GPB2 algorithms are negligible. To our present comparison the other 12 cases (1, 2, 3, 4, 7, 9, 10, 11, 12, 13, 14, 15) are interesting. For each of these 12 cases we simulated the GPB1, the GPB2, and the IMM algorithms and ran Monte Carlo simulations, consisting of 100 runs from $t = 0$ to $t = 100$. For simplicity of interpretation of the results we used one fixed path of $\theta$ during all runs: $\theta = 0$ on the time interval [0, 30], $\theta = 1$ on the interval [31, 60], and $\theta = 0$ on the interval [61, 100].

The results of our simulations for the 12 interesting cases are as follows. In six cases (1, 2, 7, 12, 14, 15) both the IMM and the GPB2 performed slightly better than the GPB1, while the IMM and the GPB2 performed equally well. For typical results, see Fig. 2. In the other six cases both the IMM and the GPB2 performed significantly better than the GPB1. For typical results see Figs. 3 and 4. Of these six cases the IMM and the GPB2 performed four times equally well (cases 3, 4, 11, and 13) and two times significantly different (cases 9 and 10).

On the basis of these simulations we can conclude that the IMM performs almost as well as the GPB2, while its computational load is about that of GPB1. We can further differentiate this overall conclusion.

- Increasing the parameters $\tau_0$ and $\tau_1$ increases the difference in performance between GPB1 and GPB2, but not between IMM and GPB2.
- If $a$ is being switched, then the IMM performs as well as the GPB2, while the GPB1 sometimes stays significantly behind.
- If the white noise gains, $b$ or $g$, are being switched, then the IMM performs as well as the GPB2, while the GPB1 sometimes stays significantly behind.
- If only $h$ is being switched, then in some cases the IMM, and even more often, the GPB1 tend to diverge while the GPB2 works well.

Another interesting question is how the IMM compares to the modified MM algorithm and the MGEK filter. Apart from the GPB algorithms, Westwood [18] also evaluated four more filters, the MM, the modified MM, the MGEK, and a MGEK with a "postprocessor." For the 19 cases there was only one algorithm that outperformed the GPB1 algorithm in some cases. It was the MGEK filter in the cases 1, 3, and 4. He also found that the MGEK filter performed in these cases marginally or significantly less good than the GPB2 algorithm. As the above experiments showed that for cases 1, 3, and 4 the GPB2 and the IMM algorithm performed equally well, one can conclude that the MM, the modified MM, the MGEK, the MGEK with "postprocessor," and the GPB1 are in all 19 cases outperformed by the IMM algorithm.

On the basis of these comparisons one can conclude that for practical filtering applications with $N = 2$, the IMM algorithm is the best first choice. As the IMM algorithm has been developed on the basis of some general hypotheses reduction principles, which are $N$-invariant, one can reasonably expect that this is also true for larger $N$. But it is unlikely that the IMM performs in all applications almost as good as the exact filter.

TABLE I
THE PARAMETERS OF THE 19 CASES OF WESTWOOD [18]

| CASE | H-VALUES | | θ-DEPENDENT VALUES | | | |
|---|---|---|---|---|---|---|
| # | $T_0$ | $T_1$ | a(0), a(1) | b(0), b(1) | h(0), h(1) | g(0), g(1) |
| 1 | 40 | 20 | .995,.990 | 1.0 | 1.0 | 1.0 |
| 2 | 40 | 20 | .995,.990 | .5 | 1.0 | .5 |
| 3 | 40 | 20 | .995,.990 | .1 | 1.0 | 5.0 |
| 4 | 200 | 100 | .995,.990 | .1 | 1.0 | 5.0 |
| 5 | 40 | 20 | .995,.990 | 8.0 | 1.0 | 1.0 |
| 6 | 40 | 20 | .995,.990 | 1.0 | 1.0 | .3 |
| 7 | 40 | 20 | .995,.900 | .5 | 1.0 | 2.0 |
| 8 | 40 | 20 | .995,.750 | 1.0 | 1.0 | .6 |
| 9 | 40 | 20 | 995 | 2.0 | 1.0,.95 | .5 |
| 10 | 40 | 20 | 995 | 1.0 | 1.0,.80 | .2 |
| 11 | 40 | 20 | 995 | .5 | 1.0,.80 | .8 |
| 12 | 4 | 2 | 995 | .5 | 1.0,.80 | .8 |
| 13 | 200 | 100 | 995 | .5 | 1.0,.80 | .8 |
| 14 | 40 | 20 | 995 | 1,5.0 | 1.0 | 1.0 |
| 15 | 40 | 20 | 995 | 1.0 | 1.0 | 1,5.0 |
| 16 | 10 | 2 | 95 | .5 | 1.0,0.0 | 1.0,2.0 |
| 17 | 200 | 5 | 950,0.0 | 1.0 | 1.0 | 1.0 |
| 18 | 50 | 5 | 950,1.2 | 1.0 | 1.0 | 1.0 |
| 19 | 10 | 2 | .95 | .5 | 1.0 | 1.0,40.0 |

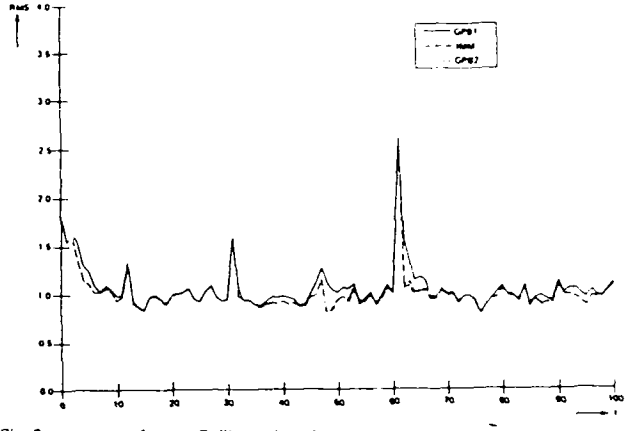

Fig. 2. rms error for case 7, illustrative of the six cases (1, 2, 7, 12, 14, 15) where both IMM and GPB2 perform slightly better than GPB1.

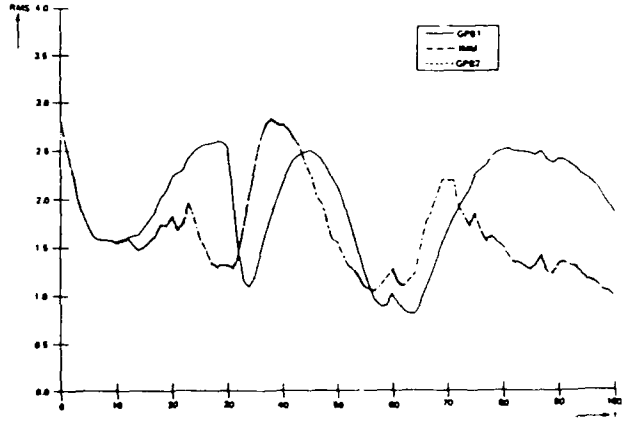

Fig. 3. rms error for case 3, illustrative of the four cases (3, 4, 11, 13) where both IMM and GPB2 perform better than GPB1, while IMM and GPB2 perform equally well.

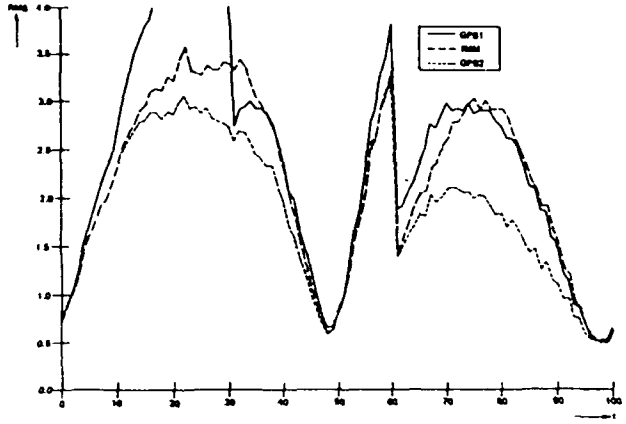

Fig. 4. rms error for case 9, illustrative of the two cases (9 and 10) where IMM performs better than GPB1, but slightly worse than GPB2 (in these two cases only $h$ jumps).

Therefore, if the IMM performs not well enough in a particular application one should consider using a suitable GPB ($\geq 2$) or DE algorithm [14], or one might try to design a better algorithm by using adaptive merging techniques [16]. The DE algorithm might possibly be improved by the novel timing of hypotheses reduction [1]. If for a particular application the performance of the selected algorithm has a too high computational load, then it is best to try to exploit some geometrical structure of the problem considered [2], [11].

In situations where estimation has to be done outside some time-critical control loop, it is usually preferable to use a smoothing algorithm instead of a filtering algorithm [8], [14], [21]. In view of the above filtering results, this suggests that the ideas that underly the IMM algorithm can be exploited to develop better smoothing algorithms.

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# Failure Detection Via Recursive Estimation for a Class of Semi-Markov Switching Systems

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## Abstract

An area of current interest is the estimation of the state of discrete-time stochastic systems with parameters which may switch among a finite set of values. The parameter switching process of interest is modeled by a class of semi-Markov chains. This class of processes is useful in that it pertains to many areas of interests such as the failure detection problem, the target tracking problem, socio-economic problems and in the problem of approximating nonlinear systems by a set of linearized models. It is shown in this paper how the transition probabilities, which govern the model switching at each time step, can be inferred via the evaluation of the conditional distribution of the sojourn time. Following this, a recursive state estimation algorithm for dynamic systems with noisy observations and changing structures, which uses the conditional sojourn time distribution, is derived.

## 1. Introduction

In this paper we are concerned with failure detection via recursive estimation of parameters in discrete-time dynamic systems. The topic of interest is stochastic systems with abruptly changing parameters i.e., model jumps. The recursive state estimation algorithm for this problem developed in this paper provides the conditional model probabilities used for detecting the change in system parameters which signify component failures.

The abruptly changing parameters, which switch among a finite set of values, are modeled as a Markov or a semi-Markov chain with known transition statistics [M2,M3,M5-M8,G1]. Although the idea of semi-Markov chains is appropriate for the model concerned, the analysis presented in the above is actually only for Markov chains (since the transition probabilities were assumed fixed and the transitions depended only on the latest state - see Eq. (8) in [M2]). The process considered in this paper is of the semi-Markov type and pertains to many areas of interest. A failure in a component of a dynamical system can be represented by a sudden change in the systems parameters [B5,S1,W1]. Also, a repair to a system represents a change in the parameters [B5]. Other areas that this class of processes pertains to are the target tracking problem [B1], socio-economic problems [G2] and the technique of approximating grossly nonlinear systems by a set of linearized models [M4,V1,V2].

The first treatment of estimation in a switching environment was in [A1] where the means and covariances of the process and measurement noises experienced jumps. As indicated in [C1], the optimum state estimation in a multiple model environment is a function of the elemental ("model-matched") state estimates obtained via estimators tuned to all possible parameter histories. Thus, with time, the estimator must keep track of an exponentially growing number of parameter history hypotheses. Even in the case of Markov switching the estimation algorithm requires exponentially growing memory [T1, T2]. Suboptimal algorithms like the Generalized Pseudo-Bayesian Algorithm (GPB) [A1, C1, T2] and the Interacting Multiple-Model Algorithm (IMM) [B2, B3, B4] are viable approaches to obtain a real-time implementable estimation algorithm. These algorithms rely on different hypothesis merging techniques to limit the memory and computational requirements [B4].

In [S2,C2] a semi-Markov switching problem was considered, but the jumps were assumed to be perfectly observed. In [M9] an estimation scheme for semi-Markov processes was developed based upon the detection-estimation algorithm (DEA). This approach is obtained by retaining a certain number of most likely parameter history hypotheses. The estimation schemes based upon the DEA (which discards all but a number of most likely history hypotheses) and the GPB or IMM (which use hypothesis merging) algorithms represent different philosophies of algorithm design. We present an example comparing the two methods for a particular state estimation problem later in this paper.

The problem is formulated in Section 2. In Section 3 the sojourn time conditional probability mass functions and the conditional transition probabilities which we derived in [M1a], are given here for clarity and ease of reference. The inclusion of Section 4, the state estimation algorithm which was developed in [M1b], is for the sake of completeness. In Section 5 simulations are presented. Preliminary results on this problem were presented in [M1a, M1b].

## 2. *Formulation of the problem*

The system is modeled by the equations

$$x(k) = F[M(k)]\, x(k-1) + v(k-1, M(k)) \quad (2.1)$$

$$z(k) = H[M(k)]\, x(k) + w(k, M(k)) \quad (2.2)$$

where M(k) denotes the model "at time k" - in effect during the sampling period ending at k. The process and measurement noise sequences, v(k) and w(k), are white and mutually uncorrelated.

The model at time k is assumed to be among the possible r models

$$M(k) \in (1, \ldots, r) \quad (2.3)$$

For example

$$F[M(k)=j] = F_j \quad (2.4)$$

$$v(k-1, M(k)=j) \sim N(u_j, Q_j) \quad (2.5)$$

i.e., the structure of the system and/or the statistics of the noises might be different from model to model. The mean $u_j$ of the noise can model a maneuver or a system failure as a deterministic input.

The model switching process to be considered here is of the semi-Markov type. The process is specified by a family of transition matrices $p_{ij}(\tau_i)$ i.e., it is a "sojourn-time-dependent Markov" (STDM)

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chain, which belongs to the semi-Markov class. The specification of the STDM chain is more closely related to physical models because it does not have the artificial restart of the sojourn time counting of the semi-Markov process for virtual transitions* and can capture important features in many realistic situations.

For the class of semi-Markov chains governing the evolution of the system's model considered here, we need the pdf of the sojourn time conditioned on the observations, to infer the transition probabilities. The conditional transition probabilities based on noisy observations of the system's state are obtained in the next section.

*A semi-Markov (SM) chain [H1, H2, R1] is characterized by a fixed matrix of transition probabilities $[p_{ij}]$ and a matrix of sojourn time probability density functions $[f_{ij}(\tau_i)]$; which are functions of the current state $i$ as well as the destination state $j$ of the transition. In a SM chain first the destination of the jump is chosen according to $[p_{ij}]$ and then the time after which the jump takes place (i.e., the sojourn time) is chosen according to $[f_{ij}(\tau_i)]$. In this model the process can undergo a virtual transition (i.e., jump "in place" if $j=i$); however, in this case, the sojourn time counting is still restarted even though the system has been in state $i$ for some time.

## 3. Sojourn Time Probability Mass Functions and Conditional Transition Probabilities

The process $M(k)$, $k=0,1,\ldots$, which represents the system model, can exist in one of $r$ possible states. The current probabilities of transition for the STDM process (chain) are functions of the sojourn time $\tau$ and are defined as

$$p_{ij}(\tau) = P(M(k)=j|M(k-1)=i,\tau_i(k-1)=\tau) \qquad (3.1)$$

where $\tau_i(k-1)$ is the sojourn time in state $i$ at time $k-1$. It is assumed that at $k=0$ the sojourn time (in whatever state the system model is) is $\tau=1$. Thus the values $\tau$ can take are from 1 to the maximum, which at time $k-1$ is then $k$.

Let $z(i)$ be a noisy measurement of the state of the dynamic system whose model undergoes transitions according to the above described STDM process. Based on the available information $Z^k=\{z(\kappa)\}_{\kappa=1}^{k}$ the probability of the model process being in state $i$, denoted as $\mu_i(k)$, is defined as

$$\mu_i(k)\triangleq P(M(k)=i|Z^k) \qquad i=1,\ldots,r \qquad (3.2)$$

The conditional pmf of the sojourn time in state $M(k)=i$ based on the available information $Z^k$ at time $k$ is

$$g_i^k(\tau) \triangleq P(\tau_i(k)=\tau|M(k)=i,Z^k) = P(\tau_i(k)=\tau|M(k)=i,Z^{k-1})$$
$$= P(M(k-1)=i,\ldots,M(k-\tau+1)=i,M(k-\tau)\neq i|M(k)=i,Z^{k-1}) \qquad (3.3)$$

where the perfect knowledge of the state $M(k)$ allows one to go down to one index less in the conditioning, i.e., $Z^{k-1}$.

Following (3.1) the conditional probability of transition from $i$ to $j$ at time $k-1$ given the observations $Z^{k-1}$ is, in terms of (3.3),

$$\hat{p}_{ij}(k-1) \triangleq P(M(k)=j|M(k-1)=i,Z^{k-1})$$
$$= \sum_{\tau=1}^{k} P(M(k)=j|M(k-1)=i,\tau_i(k-1)=\tau,Z^{k-1}) \cdot P(\tau_i(k-1)=\tau|M(k-1)=i,Z^{k-1})$$
$$= \sum_{\tau=1}^{k} p_{ij}(\tau)\, g_i^{k-1}(\tau) \qquad (3.5)$$

Note that the argument of $p_{ij}$ defined in (3.1) is the sojourn time while the argument of $\hat{p}_{ij}$ defined above is the current time.

The conditional probability mass function (3.3) of the sojourn time $\tau$ in state $i$ at time $k$ is given by the following expressions

$$g_i^k(1) = 1 - \frac{\mu_i(k-1)}{a_i(k,1)}\, b_i(k,1) \qquad (3.6)$$

$$g_i^k(s) = \left[1 - \frac{\mu_i(k-s)}{a_i(k,s)}\, b_i(k,s)\right] \prod_{m=1}^{s-1} \frac{\mu_i(k-m)}{a_i(k,m)}\, b_i(k,m) \qquad s=2,\ldots,k \qquad (3.7)$$

$$g_i^k(k+1) = \prod_{m=1}^{k} \frac{\mu_i(k-m)}{a_i(k,m)}\, b_i(k,m) \qquad (3.8)$$

Expressions (3.6)-(3.8) are proven by induction in [M1a]. The notations $a_i$ and $b_i$ used above are defined below.

The probability that the process will stay $s$ time steps in the same state $i$ as it is at time $k-s$ is, conditioned on the information at $k-s$, given by the expression

$$b_i(k,s) \triangleq P(M(k)=i,\ldots,M(k-s+1)=i|M(k-s)=i,Z^{k-s})$$
$$= \sum_{n=1}^{k-s+1} \prod_{j=n}^{n+s-1} p_{ii}(j)\, g_i^{k-s}(n) \qquad s=1,\ldots,k \qquad (3.9)$$

Conditioned on the available information $Z^{k-s}$ at time $k-s$, the joint probability of the process residing in the same state $i$ for the next $s$ time steps is denoted as

$$a_i(k,s)\triangleq P(M(k)=i,\ldots,M(k-s+1)=i|Z^{k-s})$$
$$= \sum_{j=1}^{r} P(M(k)=i,\ldots,M(k-s+1)=i|M(k-s)=j,Z^{k-s})P(M(k-s)=j|Z^{k-s})$$
$$= b_i(k,s)\mu_i(k-s) + \sum_{j\neq i} P(M(k)=i,\ldots,M(k-s+1)=i|M(k-s)=j,Z^{k-s}) \cdot \mu_j(k-s)$$
$$= b_i(k,s)\,\mu_i(k-s) + \sum_{j\neq i} \left[\sum_{n=1}^{k-s+1} P(M(k)=i,\ldots,M(k-s+1)=i|M(k-s)=j,\tau_j(k-s)=n,Z^{k-s}) \cdot g_j^{k-s}(n)\right]\mu_j(k-s)$$
$$= b_i(k,s)\,\mu_i(k-s) + \sum_{j\neq i} \left[\sum_{n=1}^{k-s+1} p_{ji}(n)p_{ii}(1)p_{ii}(2)\ldots p_{ii}(s-1)\, g_j^{k-s}(n)\right]\mu_j(k-s)$$
$$= b_i(k,s)\,\mu_i(k-s) + \sum_{j\neq i} \left[\sum_{n=1}^{k-s+1} p_{ji}(n)\prod_{l=1}^{s-1} p_{ii}(l)\, g_j^{k-s}(n)\right]\mu_j(k-s) \qquad s=1,\ldots,k \qquad (3.10)$$

## 4. The state estimation algorithm

As indicated in Sec. 1, the optimal estimator for linear systems with Markov model jumps requires an exponentially increasing memory. Among the suboptimal approaches discussed, it appears that the IMM is the most cost-effective in implementation [B4]. In view of this, the state estimation for a linear system with sojourn-time-dependent transition probabilities is developed in the sequel based on the IMM approach.

In this approach, at time $k$ the state estimation is computed under each possible model hypothesis using $r$ filters (for the $r$ possible models), with each filter using a different combination of the previous model-conditioned estimates. Each model transition probability is a known function of the sojourn time given by (3-1). Each model has a sojourn time $\tau_i(k)$ in state $i$ which is, however, not known. The filter has access only to the observations from which the conditional pmf of the sojourn time (3.6) - (3.8) can be obtained.

this in turn is to be used in calculation of the conditional transition probabvilities (3.5)

To find the conditional pdf of the state of the dynamic system described by (2.1)-(2.3) the total probability theorem is used as follows:

$$p[x(k)|Z^k] = \sum_{j=1}^{r} p[x(k)|M(k)=j,z(k),Z^{k-1}]\ P[M(k)=j|Z^k]$$

$$= \sum_{j=1}^{r} p[x(k)|M(k)=j,z(k),Z^{k-1}]\ \mu_j(k) \qquad (4.1)$$

i.e., $r$ filters running in parallel. The model-conditioned posterior pdf of the state, can be rewritten as (with the irrelevant conditioning on $Z^{k-1}$ in the numerator omitted)

$$p[x(k)|M(k)=j,z(k),Z^{k-1}]$$

$$= \frac{p[z(k)|M(k)=j,x(k)]}{p[z(k)|M(k)=j,Z^{k-1}]}\ p[x(k)|M(k)=j,Z^{k-1}] \qquad (4.2)$$

reflecting one cycle of the state estimation filter matched to model $j$ starting with the prior, which is the last term above. The total probability theorem is now applied to this prior, yielding

$$p[x(k)|M(k)=j,Z^{k-1}]$$

$$= \sum_{i=1}^{r} p[x(k)|M(k)=j,M(k-1)=i,Z^{k-1}]P(M(k-1)=i|M(k)=j,Z^{k-1})$$

$$= \sum_{i=1}^{r} p[x(k)|M(k)=j,M(k-1)=i,Z^{k-1}]\ \mu_{i|j}(k-1|k-1) \qquad (4.3)$$

where

$$\mu_j(k) \triangleq P(M(k)=j|Z^k) \qquad (4.4)$$

and

$$\mu_{i|j}(k-1|k-1) \triangleq P(M(k-1)=i|M(k)=j,Z^{k-1}) \qquad (4.5)$$

Note that Eq. (4.3) represents a Gaussian mixture under the typical Gaussian assumptions on the noise terms in Eqs. (2.1) and (2.2). This mixture is then approximated by a single moment-matched Gaussian.*

*Therefore it follows* that the input to the filter matched to model $j$, $j=1,\ldots,r$, is obtained from an interaction of these $r$ filters. This interaction consists of the mixing of the estimates $\hat{x}^i(k-1|k-1)$ according to the weightings (probabilities) $\mu_{i|j}(k-1|k-1)$. The evaluation of the probabilities (4.4) and (4.5) in the STOM situation, are the key results needed to obtain a recursive state estimation algorithm for this type of model switching. These probabilities are shown below to follow from the results in Section 3.

Fig. 4.1 describes the resulting Interacting Multiple Model (IMM) algorithm, which consists of $r$ interacting filters operating in parallel. The mixing is done at the input of the filters with the probabilities, detailed later in (4.7), conditioned on $Z^{k-1}$.

One cycle of the algorithm consists of the following:

Starting with the model-conditioned estimate $\hat{x}^i(k-1|k-1)$, with associated covariance $P^i(k-1|k-1)$, one computes the mixed initial condition for the filter matched to $M(k)=j$ according to (4.3) as follows

$$\hat{x}^{0j}(k-1|k-1) = \sum_{i=1}^{r} \hat{x}^i(k-1|k-1)\mu_{i|j}(k-1|k-1) \qquad (4.6)$$

From (4.5)

$$\mu_{i|j}(k-1|k-1) = \frac{1}{\bar{c}_j}P(M(k)=j|M(k-1)=i,Z^{k-1})P(M(k-1)=i|Z^{k-1})$$

$$= \frac{1}{\bar{c}_j}\ \hat{p}_{ij}(k-1)\ \mu_i(k-1) \qquad (4.7)$$

*This is the key step of the IMM that yields an algorithm with fixed (and modest) computational requirements: using $r$ filters it yields performance comparable to the Generalized Pseudo Bayesian algorithm with $r^2$ filters [B4].

where the notations from (4.4) and (3.5) were used and

$$\hat{x}^i(k-1|k-1) \triangleq E[x(k-1)|M(k-1)=i,Z^{k-1}] \qquad (4.8)$$

is the model-conditioned state estimate at time k-1. The expression of $\hat{p}_{ij}$ for the STOM case using terms involving sojourn time probabilities is the one obtained in (3.5). The covariance corresponding to (4.6) is

$$P^{0j}(k-1|k-1) = \sum_{i=1}^{r} \mu_{i|j}(k-1|k-1)\{P^i(k-1|k-1)$$

$$+ [\hat{x}^i(k-1|k-1)-\hat{x}^{0j}(k-1|k-1)]$$

$$\cdot[\hat{x}^i(k-1|k-1)-\hat{x}^{0j}(k-1|k-1)]'\} \qquad (4.9)$$

The estimate (4.6) and covariance (4.9) are used as input to a standard Kalman filter matched to $M(k)=j$ to yield the model-conditioned estimate $\hat{x}^j(k|k)$ and its covariance $P^j(k|k)$.

The likelihood functions corresponding to the $r$ filters are computed as

$$\Lambda_j(k) = p[z(k)|M(k)=j,Z^{k-1}]$$

$$\approx p[z(k)|M(k)=j,\hat{x}^{0j}(k-1|k-1),P^{0j}(k-1|k-1)] \qquad (4.10)$$

where the past data have been replaced by (4.6) and (4.8) according to the key step of the IMM. The model probabilities (4.4) are updated as follows:

$$\mu_j(k) = P(M(k)=j|Z^k) = \frac{1}{\bar{c}}\Lambda_j(k)\sum_{i=1}^{r} \hat{p}_{ij}(k-1)\ \mu_i(k-1) \qquad (4.11)$$

where the conditional transition probabilities, $\hat{p}_{ij}$, are as given in (4.8).

Eqs (4.7) and (4.11) in combination with $\hat{p}_{ij}$ are the key results that make possible the state estimation for a system with sojourn-time-dependent model transitions.

Finally, for output only, the latest state estimate and covariance are obtained according to Eqs (4.1) and (4.3) as

$$\hat{x}(k|k) = \sum_{j=1}^{r} \hat{x}^j(k|k)\ \mu_j(k) \qquad (4.12)$$

$$P(k|k) = \sum_{j=1}^{r} \mu_j(k)\{P^j(k|k)$$

$$+ [\hat{x}^j(k|k) - \hat{x}(k|k)][\hat{x}^j(k|k) - \hat{x}(k|k)]'\} \qquad (4.13)$$

## 5. Simulation Results

The algorithm developed in Sec. 4 using the sojourn time pmf obtained in Sec. 3 is used to estimate the state of the system. In the first example the results of this STOM-based IMM estimation scheme are compared with results obtained from an IMM algorithm based upon a Markov model transition assumption. In the second example the STOM-based IMM estimation scheme is compared to the detection-estimation algorithm of [M9]. It is assumed that an STOM process described in Sec. 2 governs the switching between models. In the following T is the sampling period and k is an integer representing the number of sampling periods since time zero.

**Example 1**

The estimation of a controlled double integrator system with process and measurement noises is considered with a gain failure. The two possible models are given by the following system equation

$$x^i(k+1) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x^i(k)$$

$$+ \begin{bmatrix} 0 \\ b^i \end{bmatrix} u(k) + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} v(k) \qquad i=1,2 \quad (5.1)$$

with measurement equation

$$z(k) = [1\ 0]\ x^i(k) + w(k) \qquad (5.2)$$

The models differ in the control gain parameter $b^i$. The process and measurement noises are mutually uncorrelated with zero mean and variances given by

$$E[v(k)\ v(j)] = 4\cdot10^{-2}\ \delta_{kj} \qquad (5.3)$$

and

$$E[w(k)\, w(j)] = \delta_{kj} \qquad (5.4)$$

The control gain parameters were chosen to be $b^1=2$ and $b^2=1$.

The transition probabilities $p_{11}(\tau)$ and $p_{22}(\tau)$ defined in (3-1) are shown in Fig 5-1. Note that $p_{ij}(\tau)$, for $i \neq j$, are given by

$$p_{ij}(\tau) = 1 - p_{ii}(\tau) \qquad (5.5)$$

Thus we see that $p_{11}(\tau)$ is initially .5 and rises rapidly to .99 and then decreases towards .1 which is its steady state value. We also see that $p_{22}(\tau)$ has a value close to 1.0 for this range of $\tau$ and thus model state two is essentially an absorbing state.

Figs. 5-2 through 5-4 present the results of 100 Monte Carlo runs. The true system was initially model 1 for every run and the model transitions occurred according to the probabilities of Fig. 5-1. For simplicity, since we are mainly interested in the estimation of the state, and not in the control strategy, we set $u(k)=3$ for all $k$.

The Markov based IMM used for comparison utilized the a priori average transition probabilities $\bar{p}_{ii}(\tau)$, obtained by taking the expected value of the transition probabilities shown in Fig. 5-1. In other words, the conditional probability $\hat{p}$ from (3.5) is replaced by the a priori (unconditional) $\bar{p}$ given below in (5.7). The probability of having a sojourn time $\tau_i$ equal to $\tau$ is the probability that model $i$ is in effect for $\tau-1$ steps, and then a transition occurs at step $\tau$.

$$P(\tau_i=\tau) = \left[\prod_{t=1}^{\tau-1} p_{ii}(t)\right]\left[1 - p_{ii}(\tau)\right] \qquad (5.6)$$

Thus we get

$$\bar{p}_{ii} = \sum_{\tau=1}^{\infty} p_{ii}(\tau)\, P(\tau_i=\tau) \qquad i=1,2 \qquad (5.7a)$$

and

$$\bar{p}_{ij} = 1 - \bar{p}_{ii} \qquad (5.7b)$$

Figs. 5-2 and 5-3 are plots of the RMS error in $x_1(k)$ and $x_2(k)$ respectively. From Fig. 5-2 we can see that the STDM-based IMM estimator improves the RMS error in $x_1(k)$ by as much as 20 percent. From Fig. 5-3 we see that the RMS error in $x_2(k)$ of the STDM-based IMM estimator is as low as one third the error of the Markov-based IMM scheme. Thus the mean-square error improved by *an order of magnitude*.

Fig. 5-4 is a plot of the average model probability error. This is the error in the filter's determination of the correct system model.

Typical running times for the STDM-based IMM vs. the Markov-based IMM are in the ratio of 3:1. The length of the time-span over which the sojourn time pmf is computed can be truncated - it becomes negligible after 15 steps. This keeps within reasonable limits the additional calculations of the STDM-based filter and prevents any growth of the computational or memory requirements.

**Example 2**

In this example we make a comparison between the detection-estimation algorithm, (DEA), based semi-Markov estimator of [M9] with the STDM-based IMM estimator of this paper. For this purpose the system and the semi-Markov model switching process attributes are as in [M9] example 3, and are repeated here for ease of reference.

The model process $M(k)$ is taken as a semi-Markov chain. The scalar system is described by [M9]

$$x(k+1) = 1.04\, x(k) + v(k)$$
$$z(k) = x(k) + D(M(k))w(k), \qquad K=0,1,2,\ldots \qquad (5.8)$$

where $r = 3$ models, $D(1)=100$, $D(2)=10$, and $D(3)=1$. Here $\{v(k)\}$ and $\{w(k)\}$ are mutually independent zero-mean Gaussian white noise sequences with covariances $Q=0.1$ and $R=1.0$, respectively. The initial conditions are $x(0)\sim N(30,400)$, $P\{M(0)=i\}=1/3$ for $i=1,2,3$. For the real system $x(0)=1$ in every simulation. The process $M(k)$ is modeled by a semi-Markov chain with the imbedded Markov chain transition probabilities given by $p_{11}=p_{22}=p_{33}=0$, $p_{12}=0.7$, $p_{13}=0.3$, $p_{21}=0.6$, $p_{23}=0.4$, $p_{31}=0.3$, and $p_{32}=0.7$. The sojourn time probability mass functions $p_i(\tau)$ are assumed to be

$$p_1(\tau) = a_1 \exp[-|\tau-3|]$$
$$p_2(\tau) = a_2 \exp[-|\tau-6|]$$
$$p_3(\tau) = a_3 \exp[-|\tau-8|] \qquad (5.9)$$

for $\tau \geq 0$ with $a_i$ such that

$$\sum_{\tau=0}^{\infty} p_i(\tau) = 1, \quad i=1,2,3 \qquad (5.10)$$

The results of 50 Monte Carlo runs average are shown in Figs. 5-5, 5-6. In Fig. 5-5 we compare the rms state errors of the two filter DEA based semi-Markov estimator of [M9] with our two filter GPB based semi-Markov approach, and with the GPB estimator using 3 filters. Note that the values for the DEA estimator are two-time-step smoothed values (see [M9], Fig. 7, M=2 most likely histories retained) whereas the values for the STDM-IMM estimator are filtered values. We can see that our estimator with two filters is stable as opposed to the unstable two-filter DEA method.

The plot of the 3 filter STDM-IMM estimator shown in Fig. 5-5 is given so that one can compare the improvement obtainable by adding an extra filter to this approach. We see that the long term trend is for the 3 filter STDM-IMM to give a smaller rms error than the version with 2 filters.

In Fig. 5-6 we compare the probability of error obtained using a 4 filter DEA estimator versus the 3 filter STDM-IMM estimator. Both curves were obtained from a filtering operation (see [M9] Fig. 10, N=0). We can see that the present estimator gives a much clearer indication of the correct system structure and hence is preferable for failure detection.

## 6. Conclusion

We have applied the recursive state estimation algorithm for dynamic systems, whose state model experiences jumps according to a sojourn-time-dependent Markov, STDM, chain, to the problem of failure detection. The algorithm, which is of the IMM type, uses noisy state observations and the calculations are done in the following order:

1. Probability of each model being the current model
2. Sojourn time pmf in the current model
3. Model-conditioned state vector estimates and covariances
4. Overall state vector estimate and its covariance.

The first example simulated indicates that the use of the STDM-based IMM estimator can give a substantial improvement in state estimation over a Markov-based IMM. The latter relies on the a priori average transition probabilities while the former uses conditional transition probabilities obtained from the conditional sojourn time distribution. This example shows that the STDM-based scheme is substantially better than the Markov-based scheme in determining the true system model, which is beneficial for failure detection schemes.

The second example simulated shows that, for the particular system under consideration the STDM-based

IMM estimator, which is an hypothesis merging technique, compares favorably in terms of the probability of error, to the detection-estimation algorithm based estimator, which discards the unlikely parameter history hypothesis

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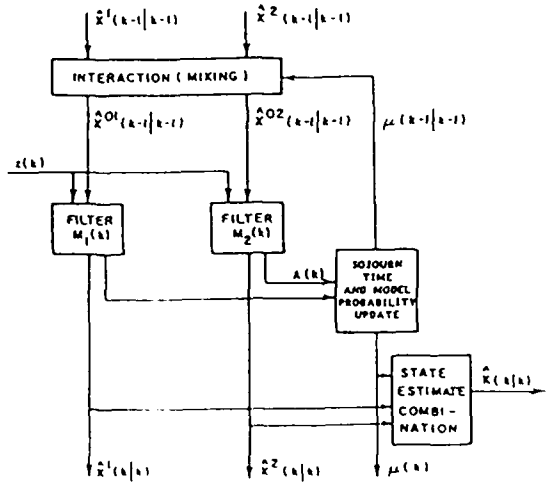


Figure 4.1 The IMM algorithm

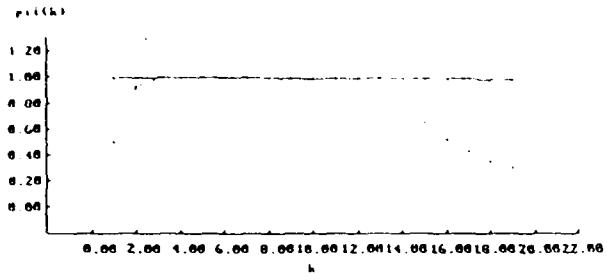


Figure 5-1 Transition probabilities $p_{11}(k)$ and $p_{22}(k)$ $p_{11}(k)$ is shown by the dotted line

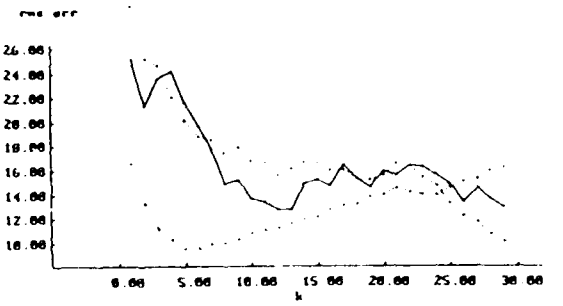


Figure 5-5 RMS state errors for example 2 Note that IMM2, stands for a two-filter STOM-IMM based estimator etc

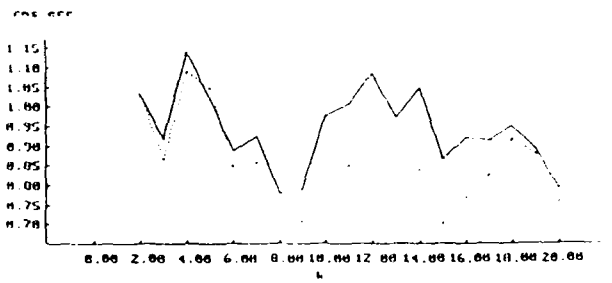


Figure 5-2 RMS error in $x_1(k)$, STOM-based IMM and Markov-based IMM Markov-based IMM errors are shown by the solid line

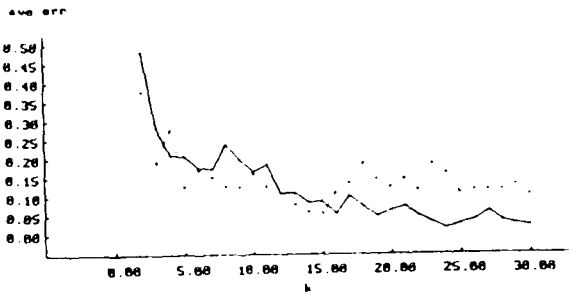


Figure 5-6 Probability of error in system structure detection for example 2 Note that DEA4 stands for the 4-filter DEA based estimator etc

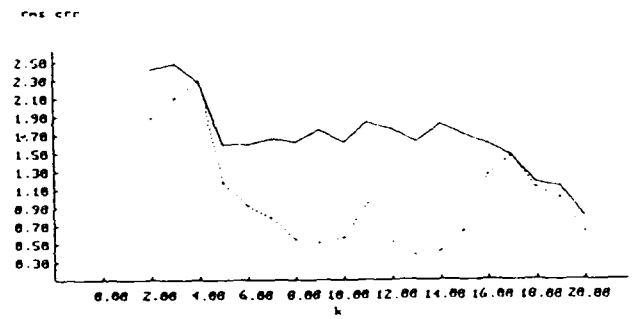


Figure 5-3. RMS error in $x_2(k)$, STOM-based IMM and Markov-based IMM Markov-based IMM errors are shown by the solid line

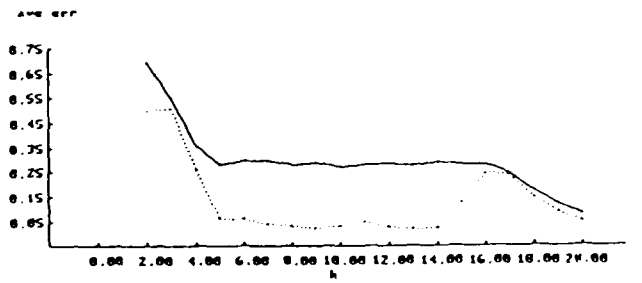


Figure 5-4. Average model probability error magnitudes, STOM-based IMM and Markov-based IMM Markov based IMM errors are shown by the solid line

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# Distributed Adaptive Estimation with Probabilistic Data Association*

K. C. CHANG† and Y. BAR-SHALOM‡§

*A fusion algorithm for target state estimation under cluttered environment with uncertain measurement origins and uncertain system models in a distributed manner can be applied for tracking a maneuvering target in a cluttered and low detection environment*

**Key Words**—Distributed estimation; multiple model; target tracking; probabilistic data association; Bayesian methods; distributed sensor networks.

**Abstract**—The probabilistic data association filter (PDAF) estimates the state of a target in a cluttered environment. This suboptimal Bayesian approach assumes that the exact target and measurement models are known. However, in most practical applications, there are difficulties in obtaining an exact mathematical model of the physical process. In this paper, the problem of estimating target states with uncertain measurement origins *and* uncertain system models in a *distributed* manner is considered. First, a scheme is described for local processing, then the fusion algorithm which combines the local processed results into a global one is derived. The algorithm can be applied for tracking a maneuvering target in a cluttered and low detection environment with a distributed sensor network.

## 1. INTRODUCTION

THE MAJOR difficulty in tracking a target with switching models/parameters in a cluttered environment is due to the fundamental conflict between the operations of model/parameter identification and data association, since the measurements with large innovations are considered as unlikely to have originated from the target of interest. In this paper, a multiple model approach in conjunction with the probabilistic data association (PDA) filter (Bar-Shalom and Tse, 1975; Bar-Shalom, 1978) to track a target with switching models using distributed sensors, is presented.

Several approaches have been proposed to perform the state estimation of a system together with identification of each model (out of a finite set) in a centralized framework. One of the significant schemes is the so-called generalized pseudo Bayes (GPB) method (Tugnait, 1982; Chang and Athans, 1978) and the other is the interacting multiple model (IMM) algorithm (Blom, 1984; Blom and Bar-Shalom, 1988). The general structure of these algorithms consists of a bank of filters for the state cooperating with a filter for the parameters. A GPB algorithm of order $n$ (GPB$n$) needs $N^n$ filters in its bank (Tugnait, 1982). The IMM algorithm performs nearly as well as the GPB2 method with notably less computation, namely, at the cost of GPB1 (Blom and Bar-Shalom, 1988). A distributed estimation scheme with uncertain models has also been derived (Chang and Bar-Shalom, 1987). However, in all the above approaches, a perfect data association was assumed, i.e. there is no uncertainty in measurement origins.

To take into account the data association problem, an adaptive PDA algorithm was presented in Gauvrit (1984) for tracking in a cluttered environment with unknown noise statistics. This algorithm identifies on line the unknown variances of the process and measurement noises but uses an earlier (static) multiple model approach (Bar-Shalom, 1988). In this paper, a *distributed* estimation problem which takes into account both *model* and *measurement origin* uncertainties will be derived. To handle the model uncertainty, a more general formulation with dynamic multiple models described by Markovian parameters will be adopted. These parameters may switch within a finite set of values which represent different system models. To take care of the missing and false

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measurements, the PDA scheme will be employed. The probabilities of associating measurements to a target given different system models will be computed and used to weight the combination of state estimates.

The problem is formulated in Section 2. A centralized algorithm which combines the IMM algorithm and the PDA filter, resulting in the MMPDA (multiple model PDA) filter, for local processing will be described in Section 3.* Then the fusion algorithm which combines the local processed results from multiple sensors into a global one will be presented in Section 4.

The algorithm can be applied for tracking a maneuvering target in a cluttered and low detection environment with a distributed sensor network (DSN).

## 2. PROBLEM FORMULATION

Let us consider the two-node scenario similar to that given in Chang *et al.* (1986), where each node processes the local measurements from its own sensor and sends the local estimates to the fusion processor periodically. The fusion processor then sends back the processed results after each communication time.

The dynamics of the target in track are modeled as

$$x(k) = f[x(k-1), M(k), v[M(k), k-1]] \quad (1)$$

where $x(k)$ is the state vector, $v[M(k), k-1]$ the process noise vector and $M(k)$ the system model from time $k-1$ to $k$. Assume the random model process $M(k)$ is Markov and it can only take values from a finite set $\mathbf{M}$, which contains $r$ distinct models,† i.e.

$$\mathbf{M} = \{M_j\}_{j=1}^{r}. \quad (2)$$

The measurement system is modeled as follows. If the measurement originates from the target in track, then

$$z^i(k) = h^i[x(k), M(k)] + w^i[M(k), k] \quad (3)$$

where $z^i(k)$ is the measurement vector from sensor $i$ and $w^i[M(k), k]$ is the corresponding measurement noise vector. The two noise sequences are mutually independent and independent of the initial state.

As in the PDA filter, it is assumed that a rule of validation of the candidate measurements‡ is available such that it guarantees that the current return will be retained with a given probability. For each sensor, denote the validated measurements at time $k$ as

$$Z^i(k) = \{z_j^i(k)\}_{j=1}^{m_k^i} \quad (4)$$

where $m_k^i$ is the number of validated measurements of sensor $i$ at time $k$, and

$$Z^{i,k} \triangleq \{Z^i(l)\}_{l=1}^{k}. \quad (5)$$

The local model-conditioned state pdfs at sensor $i$ are

$$p(x(k) \mid M_j(k), Z^{i,k}, Y^{i,k}), \quad i = 1, 2; \quad j = 1, \ldots, r \quad (6)$$

with the corresponding model probabilities

$$P\{M_j(k) \mid Z^{i,k}, Y^{i,k}\}, \quad i = 1, 2; \quad j = 1, \ldots, r \quad (7)$$

where

$$Y^{i,k} = \{Y^i(1), \ldots, Y^i(k)\} \quad (8)$$

and $Y^i(k)$ denotes the information received by node $i$ during the sampling period ending at time $k$, which is defined as the fusion result (namely, global conditional pdf) up to time $k-1$.

Assuming lossless communication and that the information communicated is the sufficient statistics, i.e. the information contained in $Y^{i,k}$ is equivalent to the information in $Z^{\bar{i},k-1}$, then we have the following equality:

$$p(x(k) \mid Z^{i,k-1}, Y^{i,k}) = p(x(k) \mid Z^{i,k-1}, Z^{\bar{i},k-1}) = p(x(k) \mid Z^{k-1}) \quad (9)$$

where $\bar{i}$ represents all sensors other than sensor $i$ and $Z^k = \{Z(l)\}_{l=1}^{k}$, where $Z(l)$ represents measurements from all sensors at time $l$.

Given the above models, the question now is how the global conditional pdf can be constructed by fusing together the local ones. Specifically, we shall investigate what is the necessary and sufficient information that has to be transmitted between nodes. The derivations will be carried out for arbitrary pdfs; however, the simulations assume linear models with Gaussian random variables, in which case the state's model-conditioned pdf (6) is Gaussian and the overall conditional pdf of the state is a Gaussian mixture (Bar-Shalom, 1988).

## 3. CENTRALIZED ALGORITHM FOR LOCAL PROCESSING

For each local node, the centralized algorithm where all measurements are sent to and processed with one processor is described below. The goal is to compute the conditional state

* The MMPDA algorithm has been implemented in the interactive software MULTIDAT (Bar-Shalom, 1987, 1988).

† The models can have states of different dimension. In this case, the lower dimension state vectors are augmented with suitable components that are zero w.p.1, to make them compatible. This is elaborated on in Section 5.

‡ Such a rule, also called "gating", considers only the measurements within some distance from the predicted measurements (for details, see, e.g. Bar-Shalom and Fortmann (1988)).

distribution given the local accumulated measurements. With only model uncertainty, the local conditional pdf at sensor $i$ can be obtained as

$$p(x(k) \mid Z^{i,k}, Y^{i,k}) = \sum_{j=1}^{r} p(x(k) \mid M_j(k), Z^{i,k}, Y^{i,k}) \times P\{M_j(k) \mid Z^{i,k}, Y^{i,k}\}. \quad (10)$$

When the additional measurement origin uncertainties are present, the above equation becomes

$$p(x(k) \mid Z^{i,k}, Y^{i,k}) = \sum_{j=1}^{r} \left\{ \sum_{\theta_{t_i}^i} p(x(k) \mid M_j(k), \theta_{t_i}^i, Z^{i,k}, Y^{i,k}) \times P\{\theta_{t_i}^i \mid M_j(k), Z^{i,k}, Y^{i,k}\} \right\} \times P\{M_j(k) \mid Z^{i,k}, Y^{i,k}\} \quad (11)$$

where $\theta_{t_i}^i$ is the event that $z_{t_i}^i(k)$ is the correct measurement and $\theta_0^i$ denotes no correct measurement.

The first term on the right-hand side of equation (11) is the standard PDA filter based on model $M_j$, where for each $\theta_{t_i}^i$

$$p(x(k) \mid M_j(k), \theta_{t_i}^i, Z^{i,k}, Y^{i,k}) = \frac{1}{c_1^i[M_j(k), \theta_{t_i}^i]} p(Z^i(k) \mid x(k), M_j(k), \theta_{t_i}^i, Z^{i,k-1}, Y^{i,k}) \times p(x(k) \mid M_j(k), Z^{i,k-1}, Y^{i,k}) \quad (12)$$

where $\theta_{t_i}^i$ has been omitted in the last term above (since it is irrelevant) and

$$c_1^i[M_j(k), \theta_{t_i}^i] = \int p(Z^i(k) \mid x(k), M_j(k), \theta_{t_i}^i, Z^{i,k-1}, Y^{i,k}) \times p(x(k) \mid M_j(k), Z^{i,k-1}, Y^{i,k})\, dx(k) = p(Z^i(k) \mid M_j(k), \theta_{t_i}^i, Z^{i,k-1}, Y^{i,k}). \quad (13)$$

Using Bayes' rule, the second term on the right-hand side of equation (11) is

$$P\{\theta_{t_i}^i \mid M_j(k), Z^{i,k}, Y^{i,k}\} = \frac{p(Z^i(k) \mid M_j(k), \theta_{t_i}^i, Z^{i,k-1}, Y^{i,k}) P\{\theta_{t_i}^i \mid M_j(k), Z^{i,k-1}, Y^{i,k}\} p(M_j(k), Z^{i,k-1}, Y^{i,k})}{p(Z^i(k) \mid M_j(k), Z^{i,k-1}, Y^{i,k}) \times p(M_j(k), Z^{i,k-1}, Y^{i,k})}$$

$$= \frac{1}{c_2^i[M_j(k)]} p(Z^i(k) \mid M_j(k), \theta_{t_i}^i, Z^{i,k-1}, Y^{i,k}) \times P\{\theta_{t_i}^i \mid M_j(k), Z^{i,k-1}, Y^{i,k}\}$$

$$= \frac{1}{c_2^i[M_j(k)]} c_1^i[M_j(k), \theta_{t_i}^i] \times P\{\theta_{t_i}^i \mid M_j(k), Z^{i,k-1}, Y^{i,k}\} \quad (14)$$

where

$$c_2^i[M_j(k)] = \sum_{\theta_{t_i}^i} c_1^i[M_j(k), \theta_{t_i}^i] \times P\{\theta_{t_i}^i \mid M_j(k), Z^{i,k-1}, Y^{i,k}\} = p(Z^i(k) \mid M_j(k), Z^{i,k-1}, Y^{i,k}). \quad (15)$$

In equation (13), the joint measurement density is (see, e.g. Bar-Shalom (1988))

$$p(Z^i(k) \mid M_j(k), \theta_{t_i}^i, Z^{i,k-1}, Y^{i,k}) = \prod_{l=1}^{m_k^i} p(z_l^i(k) \mid M_j(k), \theta_{t_i}^i, Z^{i,k-1}, Y^{i,k}) = \begin{cases} V_k^{-m_k^i} & \text{if } t_i = 0 \\ V_k^{-m_k^i+1} p[z_{t_i}^i(k) \mid M_j(k)] & \text{otherwise} \end{cases} \quad (16)$$

where $V_k$ is the volume of the validation region, because our assumption on the incorrect measurements being uniformly distributed,* independent from each other and from the correct measurement, and

$$p[z_{t_i}^i(k) \mid M_j(k)] = P_G^{-1} p(z_{t_i}^i(k) \mid M_j(k), \theta_{t_i}^i, Z^{i,k-1}, Y^{i,k}) \quad (17)$$

is the truncated density which is zero outside the validation region where $P_G$ is the probability that the correct return will lie in the validation region.

In equation (14), $P\{\theta_{t_i}^i \mid M_j(k), Z^{i,k-1}, Y^{i,k}\}$ is the prior probability of the event $\theta_{t_i}^i$ based on model $M_j$ to be correct at time $k$. By choosing a large enough validation threshold, this probability becomes independent of $M_j(k)$ and is assumed to be the same for all $\theta_{t_i}^i$ unless target signature information can be used. If no such information is available, then

$$P\{\theta_{t_i}^i \mid M_j(k), Z^{i,k-1}, Y^{i,k}\} = \begin{cases} 1 - P_G P_D & \text{if } t_i = 0 \\ \dfrac{P_G P_D}{m_k^i} & \text{otherwise} \end{cases} \quad (18)$$

where $P_D$ is the probability that the correct return will be detected.

For each model $M_j(k)$ and event $\theta_{t_i}^i$, equation (12) is the standard filtering equation. In that equation, by using the IMM approach (Blom and Bar-Shalom, 1988), the extrapolated pdf is obtained by combining the extrapolations of the

* For more elaborate models see Bar-Shalom (1988).

prior pdfs (independent of the event $\theta_{i_t}^i$)

$$p(x(k) \mid M_j(k), Z^{i,k-1}, Y^{i,k})$$
$$= \sum_{l=1}^{r} p(x(k) \mid M_j(k), M_l(k-1), Z^{i,k-1}, Y^{i,k}) \times P\{M_l(k-1) \mid M_j(k), Z^{i,k-1}, Y^{i,k}\}$$
$$\sum_{l=1}^{r} {}_{l}{}^{\prime\prime}(k) \mid M_j(k), M_l(k-1), Z^{i,k-1}, Y^{i,k})$$
$$= \frac{\times P\{M_j(k), M_l(k-1) \mid Z^{i,k-1}, Y^{i,k}\}}{P\{M_j(k) \mid Z^{i,k-1}, Y^{i,k}\}}$$
$$= \frac{1}{c_3^i[M_j(k)]} \sum_{l=1}^{r} [p(x(k) \mid M_j(k), M_l(k-1), Z^{i,k-1}, Y^{i,k}) \times P\{M_j(k) \mid M_l(k-1)\} \times P\{M_l(k-1) \mid Z^{i,k-1}, Y^{i,k}\}] \quad (19)$$

where $p(x(k) \mid M_j(k), M_l(k-1), Z^{i,k-1}, Y^{i,k})$ is the extrapolation of the conditional state pdf given $Z^{i,k-1}$ and $Y^{i,k}$ from model $M_l(k-1)$ to model $M_j(k)$ and

$$c_3^i[M_j(k)] = P\{M_j(k) \mid Z^{i,k-1}, Y^{i,k}\} = \sum_{l=1}^{r} P\{M_j(k) \mid M_l(k-1)\} \times P\{M_l(k-1) \mid Z^{i,k-1}, Y^{i,k}\}. \quad (20)$$

The last term of equation (11) is the a posteriori model probability, which is obtained as

$$P\{M_j(k) \mid Z^{i,k-1}, Y^{i,k}\} = \frac{1}{c_4^i} p(Z^i(k) \mid M_j(k), Z^{i,k-1}, Y^{i,k}) \times P\{M_j(k) \mid Z^{i,k-1}, Y^{i,k}\} = \frac{1}{c_4^i} c_2^i[M_j(k)] c_3^i[M_j(k)] \quad (21)$$

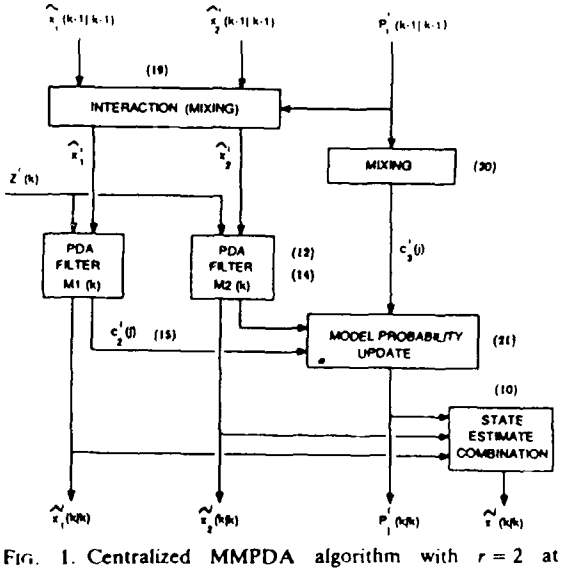


FIG. 1. Centralized MMPDA algorithm with $r = 2$ at sensor $i$.

where

$$c_4^i = \sum_{j=1}^{r} c_2^i[M_j(k)] c_3^i[M_j(k)] = p(Z^i(k) \mid Z^{i,k-1}, Y^{i,k}) \quad (22)$$

and $c_2^i[M_j(k)]$ and $c_3^i[M_j(k)]$ have been obtained in equations (15) and (20), respectively. Equations (12)–(21) complete a recursive cycle of the local processing. A flow diagram of the local MMPDA algorithm is given in Fig. 1. The flow of data is represented by the model-conditioned means $\hat{x}_j$ and the model probabilities $P_j$.

## 4. FUSION ALGORITHM

With the local conditional pdfs obtained in Section 3, we can now derive the fusion algorithm to obtain global pdf. Similar to equations (10) and (11), the global conditional pdf can be obtained as

$$p(x(k) \mid Z^k) = \sum_{j=1}^{r} p(x(k) \mid M_j(k), Z^k) P\{M_j(k) \mid Z^k\} = \sum_{j=1}^{r} \left\{ \sum_{\theta_{i_1}^1} \sum_{\theta_{i_2}^2} p(x(k) \mid M_j(k), \theta_{i_1}^1, \theta_{i_2}^2, Z^k) \times P\{\theta_{i_1}^1, \theta_{i_2}^2 \mid M_j(k), Z^k\} \right\} P\{M_j(k) \mid Z^k\} \quad (23)$$

Assuming measurements from different sensors are independent given the target state, then the first term on the right-hand side of equation (23) can be obtained as

$$p(x(k) \mid M_j(k), \theta_{i_1}^1, \theta_{i_2}^2, Z^k)$$
$$= \frac{1}{c[M_j(k), \theta_{i_1}^1, \theta_{i_2}^2]} \times p(Z(k) \mid x(k), M_j(k), \theta_{i_1}^1, \theta_{i_2}^2, Z^{k-1}) \times p(x(k) \mid M_j(k), \theta_{i_1}^1, \theta_{i_2}^2, Z^{k-1})$$
$$= \frac{1}{c[M_j(k), \theta_{i_1}^1, \theta_{i_2}^2]} \times \prod_{i=1}^{2} [p(Z^i(k) \mid x(k), M_j(k), \theta_{i_i}^i, Z^{k-1})] \times p(x(k) \mid M_j(k), Z^{k-1})$$
$$= \frac{1}{c[M_j(k), \theta_{i_1}^1, \theta_{i_2}^2]} \prod_{i=1}^{2} [p(Z^i(k) \mid x(k), M_j(k), \theta_{i_i}^i, Z^{k-1}) \times \frac{p(x(k) \mid M_j(k), Z^{k-1})]}{p(x(k) \mid M_j(k), Z^{k-1})} \quad (24)$$

where

$$c[M_j(k), \theta^1_{l_1}, \theta^2_{l_2}] = \int p(Z(k) \mid x(k), M_j(k), \theta^1_{l_1}, \theta^2_{l_2}, Z^{k-1}) \times p(x(k) \mid M_j(k), \theta^1_{l_1}, \theta^2_{l_2}, Z^{k-1})\, dx(k) = p(Z(k) \mid M_j(k), \theta^1_{l_1}, \theta^2_{l_2}, Z^{k-1}) \quad (25)$$

is the normalization constant.

Since from equations (12) and (9)

$$p(x(k) \mid M_j(k), \theta^i_{l_i}, Z^{i,k}, Y^{i,k}) = \frac{1}{c^i_1[M_j(k), \theta^i_{l_i}]} \times p(Z^i(k) \mid x(k), M_j(k), \theta^i_{l_i}, Z^{k-1}) \times p(x(k) \mid M_j(k), Z^{k-1}). \quad (26)$$

Equation (24) can be rewritten as

$$p(x(k) \mid M_j(k), \theta^1_{l_1}, \theta^2_{l_2}, Z^k) = \frac{1}{c[M_j(k), \theta^1_{l_1}, \theta^2_{l_2}]} \prod_{i=1}^{2} [c^i_1[M_j(k), \theta^i_{l_i}] \times \frac{\times p(x(k) \mid M_j(k), \theta^i_{l_i}, Z^{i,k}, Y^{i,k})]}{p(x(k) \mid M_j(k), Z^{k-1})} = \frac{1}{c_0[M_j(k), \theta^1_{l_1}, \theta^2_{l_2}]} \times \frac{\prod_{i=1}^{2} p(x(k) \mid M_j(k), \theta^i_{l_i}, Z^{i,k}, Y^{i,k})}{p(x(k) \mid M_j(k), Z^{k-1})} \quad (27)$$

where the denominator can be derived as

$$p(x(k) \mid M_j(k), Z^{k-1}) = \frac{p(x(k), M_j(k) \mid Z^{k-1})}{p(M_j(k) \mid Z^{k-1})} = \frac{\sum_{i=1}^{r} p(x(k) \mid M_j(k), M_i(k-1), Z^{k-1}) \times P\{M_j(k) \mid M_i(k-1)\} P\{M_i(k-1) \mid Z^{k-1}\}}{\sum_{i=1}^{r} P\{M_j(k) \mid M_i(k-1)\} P\{M_i(k-1) \mid Z^{k-1}\}} \quad (28)$$

and

$$c_0[M_j(k), \theta^1_{l_1}, \theta^2_{l_2}] = \frac{c[M_j(k), \theta^1_{l_1}, \theta^2_{l_2}]}{\prod_{i=1}^{2} c^i_1[M_j(k), \theta^i_{l_i}]} = \int \frac{\prod_{i=1}^{2} p(x(k) \mid M_j(k), \theta^i_{l_i}, Z^{i,k}, Y^{i,k})}{p(x(k) \mid M_j(k), Z^{k-1})}\, dx(k) \quad (29)$$

is the new normalization constant.

Assuming $\theta^1_{l_1}$ and $\theta^2_{l_2}$ are independent given the target state, then similarly to Chang *et al.* (1986), the second term of equation (23) can be obtained as

$$P\{\theta^1_{l_1}, \theta^2_{l_2} \mid M_j(k), Z^k\} = \frac{1}{c_1[M_j(k)]} \int p(\theta^1_{l_1}, \theta^2_{l_2}, Z(k) \mid x(k), M_j(k), Z^{k-1}) p(x(k) \mid M_j(k), Z^{k-1})\, dx(k) = \frac{1}{c_1[M_k(k)]} \times \int \frac{\prod_{i=1}^{2} p(x(k), \theta^i_{l_i}, Z^i(k) \mid M_j(k), Z^{k-1})}{p(x(k) \mid M_j(k), Z^{k-1})}\, dx(k) = \frac{1}{c_2[M_j(k)]} \prod_{i=1}^{2} P\{\theta^i_{l_i} \mid M_j(k), Z^i(k), Z^{k-1}\} \times \int \frac{\prod_{i=1}^{2} p(x(k) \mid M_j(k), \theta^i_{l_i}, Z^i(k), Z^{k-1})}{p(x(k) \mid M_j(k), Z^{k-1})}\, dx(k) \quad (30)$$

where

$$c_1[M_j(k)] = p(Z(k) \mid M_j(k), Z^{k-1}) \quad (31)$$

and

$$c_2[M_j(k)] = \frac{c_1[M_j(k)]}{\prod_{i=1}^{2} p(Z^i(k) \mid M_j(k), Z^{k-1})} = \frac{c_1[M_j(k)]}{\prod_{i=1}^{2} c^i_2[M_j(k)]} \quad (32)$$

are normalization constants, where $c^i_2[M_j(k)]$ was given in equation (15).

Since the information contained in $Z^{k-1}$ is the same as that in $\{Z^{i,k-1}, Y^{i,k}\}$ (see equation (9) for details), equation (30) can be written as

$$P\{\theta^1_{l_1}, \theta^2_{l_2} \mid M_j(k), Z^k\} = \frac{\prod_{i=1}^{2} P\{\theta^i_{l_i} \mid M_j(k), Z^{i,k}, Y^{i,k}\}}{c_2[M_j(k)]} \times \int \frac{\prod_{i=1}^{2} p(x(k) \mid M_j(k), \theta^i_{l_i}, Z^{i,k}, Y^{i,k})}{p(x(k) \mid M_j(k), Z^{k-1})}\, dx(k) = \frac{1}{c_2[M_j(k)]} \prod_{i=1}^{2} P\{\theta^i_{l_i} \mid M_j(k), Z^{i,k}, Y^{i,k}\} \times c_0[M_j(k), \theta^1_{l_1}, \theta^2_{l_2}]. \quad (33)$$

From equations (27) and (33), equation (23) can

be written as

$$p(x(k) \mid Z^k) = \sum_{j=1}^{r} \frac{1}{c_2[M_j(k)]} \times \left\{ \sum_{\theta_{i_1}^1} \sum_{\theta_{i_2}^2} \frac{\prod_{i=1}^{2} [p(x(k) \mid M_j(k), \theta_{i_i}^i, Z^{i,k}, Y^{i,k}) \times P\{\theta_{i_i}^i \mid M_j(k), Z^{i,k}, Y^{i,k}\}]}{p(x(k) \mid M_j(k), Z^{k-1})} \right\} \times P\{M_j(k) \mid Z^k\}. \quad (34)$$

The last term of equation (34) is the global a posteriori model probabilities. With equations (31) and (32) we have

$$\begin{aligned} P\{M_j(k) \mid Z^k\} &= \frac{1}{\bar{c}} p(Z(k) \mid M_j(k), Z^{k-1}) P\{M_j(k) \mid Z^{k-1}\} \\ &= \frac{1}{\bar{c}} c_1[M_j(k)] P\{M_j(k) \mid Z^{k-1}\} \\ &= \frac{1}{\bar{c}} \left[ c_2[M_j(k)] \prod_{i=1}^{2} c_2^i[M_j(k)] \right] P\{M_j(k) \mid Z^{k-1}\} \\ &= \frac{c_2[M_j(k)]}{\bar{c}} \times \frac{\prod_{i=1}^{2} [c_2^i[M_j(k)] P\{M_j(k) \mid Z^{k-1}\}]}{P\{M_j(k) \mid Z^{k-1}\}} \\ &= \frac{c_2[M_j(k)]}{\bar{c}} \times \frac{\prod_{i=1}^{2} [p(Z^i(k) \mid M_j(k), Z^{i,k-1}, Y^{i,k}) \times P\{M_j(k) \mid Z^{k-1}\}]}{P\{M_j(k) \mid Z^{k-1}\}} \\ &= \frac{c_2[M_j(k)]}{\bar{c}} \times \frac{\prod_{i=1}^{2} [P\{M_j(k) \mid Z^i(k), Z^{k-1}\} \times p(Z^i(k) \mid Z^{k-1})]}{P\{M_j(k) \mid Z^{k-1}\}} \\ &= \frac{c_2[M_j(k)]}{\bar{c}} \times \frac{\prod_{i=1}^{2} [P\{M_j(k) \mid Z^{i,k}, Y^{i,k}\} p(Z^i(k) \mid Z^{k-1})]}{P\{M_j(k) \mid Z^{k-1}\}} \\ &= \frac{c_2[M_j(k)]}{\bar{c}'} \frac{\prod_{i=1}^{2} P\{M_j(k) \mid Z^{i,k}, Y^{i,k}\}}{P\{M_j(k) \mid Z^{k-1}\}} \end{aligned} \quad (35)$$

where the denominator is the same as that of equation (28) and the normalization constants $\bar{c}$ and $\bar{c}'$ are

$$\bar{c} = p(Z(k) \mid Z^{k-1}) \quad (36)$$

and

$$\bar{c}' = \frac{\bar{c}}{\prod_{i=1}^{2} p(Z^i(k) \mid Z^{k-1})} = \frac{\bar{c}}{\prod_{i=1}^{2} c_4^i}. \quad (37)$$

### 4.1. *Overview of the fusion algorithm*

From the above, it follows that the global a posteriori pdf and model probabilities are obtained by combining (multiplying) the local a posteriori pdfs and model probabilities and removing (dividing) the common a priori pdf and model probabilities. From equation (34), we can see that for each model, the conditional global pdf given that this model is correct is obtained by the sum of global fused pdfs given all possible global event pairs $\theta_{i_1}^1$, $\theta_{i_2}^2$. The overall global a posteriori pdf is then obtained by the sum of global pdfs of each model weighted by the global a posteriori model probabilities. Equations (34) and (35) represent the complete cycle of fusion processing. From them it follows that the information needed to be communicated from local nodes to the fusion node consists of:

(a) the model probabilities;
(b) the association event probabilities; and
(c) the corresponding pdfs (mean and covariance for Gaussian case).

A summary flow diagram of the fusion algorithm with two models is given in Fig. 2. For

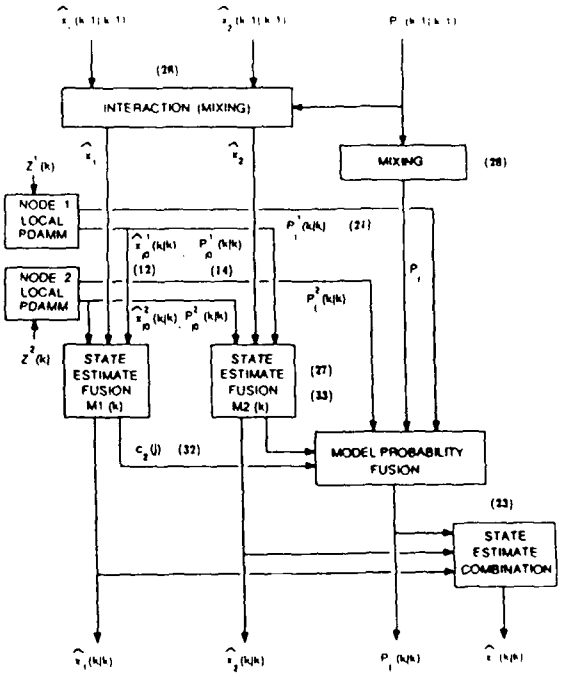


FIG. 2 Distributed MMPDA algorithm with $r = 2$.

simplicity, only the mean of each pdf is shown in the figure. References to the corresponding equations are also given in the figure.

## 5. SIMULATION RESULTS

A two-dimensional single target tracking problem will be considered. Two target dynamic models will be assumed, one with (nearly) constant velocity and the other with (nearly) constant acceleration. The Markov transition matrix of the models is known and given. The initial target state estimate is given and the initial probabilities of the two target models are assumed equal.

The target dynamic models with discretization over time intervals of length $T$ are

$$x(k) = F[M(k)]\,x(k-1) + G[M(k)]v(k-1) \quad (38)$$

where for model 1, the nearly constant velocity model, the state is

$$x = [x \quad \dot{x} \quad y \quad \dot{y}]' \quad (39)$$

and

$$F = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (40)$$

$$G = \begin{bmatrix} T^2/2 & 0 \\ T & 0 \\ 0 & T^2/2 \\ 0 & T \end{bmatrix}. \quad (41)$$

The process noise $v(k) = [v_x, v_y]'$ representing the acceleration during one period is a zero mean Gaussian white noise vector with covariance

$$\begin{bmatrix} q_{1,x} & 0 \\ 0 & q_{1,y} \end{bmatrix}.$$

For models 2 (with acceleration), the state is

$$x = [x \quad \dot{x} \quad \ddot{x} \quad y \quad \dot{y} \quad \ddot{y}]' \quad (42)$$

and

$$F = \begin{bmatrix} 1 & T & T^2/2 & 0 & 0 & 0 \\ 0 & 1 & T & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & T & T^2/2 \\ 0 & 0 & 0 & 0 & 1 & T \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (43)$$

$$G = \begin{bmatrix} T^2/2 & 0 \\ T & 0 \\ 1 & 0 \\ 0 & T^2/2 \\ 0 & T \\ 0 & 1 \end{bmatrix} \quad (44)$$

where the process noise $v(k)$ representing here the acceleration increment over one period is a zero mean Gaussian white noise vector with covariance

$$\begin{bmatrix} q_{2,x} & 0 \\ 0 & q_{2,y} \end{bmatrix}.$$

Assuming only position measurements to be available, then, for node $i$

$$z^i(k) = H^i x(k) + w_i(k) \quad (45)$$

where

$$H^i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad (46)$$

and $w^i(k)$ is a zero mean Gaussian white noise vector with covariance

$$\begin{bmatrix} r_x^i & 0 \\ 0 & r_y^i \end{bmatrix}.$$

To overcome the fact that one has different state dimensions the lower dimension vector was augmented with suitable zero components (which then have mean and variance zero) to make it compatible with the higher dimension state.

With sampling interval $T = 1$ s, the true target is simulated with constant velocity for the first seven scans, then switches to constant acceleration for the next seven scans, and finally returns to constant velocity for another seven scans. The initial target state is assumed to be $[100\,\text{m}, 30\,\text{m s}^{-1}, 0, 100\,\text{m}, 15\,\text{m s}^{-1}, 0]$ and the acceleration is assumed to be 5 and $-5\,\text{m s}^{-2}$ for the $x$ and $y$ coordinates, respectively.

The variances of the process noise are taken as $q_{1,x} = q_{1,y} = 0.1\,(\text{m s}^{-2})^2$ for model 1, the nearly constant velocity model, and $q_{2,x} = q_{2,y} = 1.0\,(\text{m s}^{-2})^2$ for model 2, the nearly constant acceleration model. The detection probabilities for both sensors are equal to 0.67 and the false alarm rates are $0.0001\,\text{m}^{-2}$. The standard deviations of the measurement errors are assumed to be $\sqrt{(10)}$ m for both $x$ and $y$ coordinates of the two sensors. The Markov transition matrix for the model parameters is assumed to be

$$\begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}.$$

The initial state estimate is generated randomly with mean the same as the true target state and covariance matrix equal to

$$\text{diag}\,[100, 1, 0.1, 100, 1, 0.1].$$

Three different configurations will be tested. First, each sensor will be simulated independently using the MMPDA algorithm described in Section 3. Second, a centralized processing with

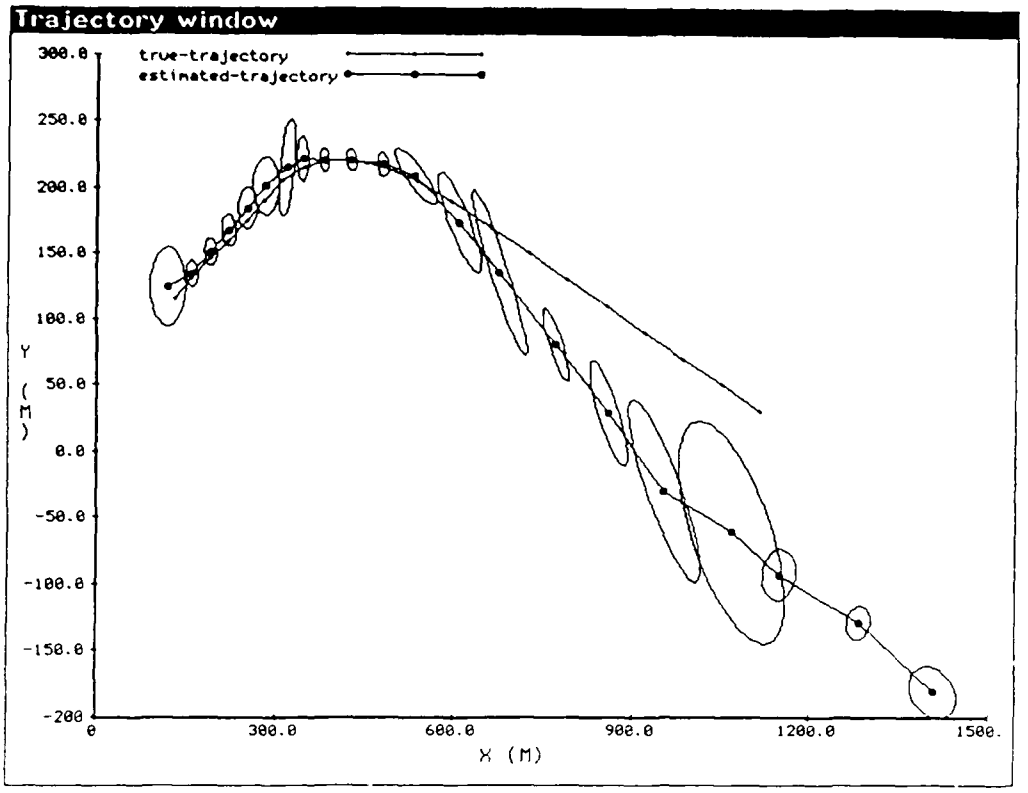


FIG. 3. Tracking results with sensor 1 only (one sample run).

measurements from both sensors will be simulated using the same MMPDA algorithm. Finally, the distributed case will be simulated. In this case, the two nodes will communicate every scan.* At each scan, each node will process its own sensor measurements first, then send the local processed results to the fusion node. After receiving the information from both local nodes, the fusion node will use the fusion algorithm derived in the previous section to construct the global estimates and send the results back to each local node.

Simulations were carried out with 50 Monte Carlo runs. The results of one sample run are shown in Figs 3–5. Figures 3 and 4 show the estimated and true trajectories of the target with sensors 1 and 2, respectively. Figure 5 shows the results for the distributed case where the two sensors interchanged their processed results. As one can see, the single sensor processed results have poor performance, and the target is lost in both cases. Figure 6 shows the probability trajectories of model 2 for the three cases as calculated by the corresponding state/model estimators. As can be seen from the figures, in both single sensor cases the algorithm fails to detect clearly the switches of the target between two models. The distributed algorithm not only responds faster in detecting the first jump of the target from the constant velocity mode to the constant acceleration mode, but also successfully detects the end of the acceleration. The centralized algorithm, which is not shown in the figures, performs exactly the same as the distributed one.

The average performances for the three configurations for 50 runs are given in Table 1. The centralized and distributed algorithms successfully track the target in 43 out of 50 runs ("successful tracking" is defined when the estimated target position is within 30 m of the true target position for the last three scans). However, out of 50 runs, sensor 1 alone and sensor 2 alone only track the target successfully in 27 and 30 runs, respectively. The r.m.s. position errors for those successful runs are also calculated. Similarly, the centralized and distributed algorithms perform better than the single sensor configurations. Note that the quality of the estimation using two sensors in terms of mean square error is significantly better than using a single sensor.

The centralized case yields the upper bound of the performance for the distributed configur-

* This is totally equivalent to the centralized configuration but has the advantages of redundancy and reliability for a DSN system. This configuration can also be used with a lower communication rate (Chang *et al.*, 1986)

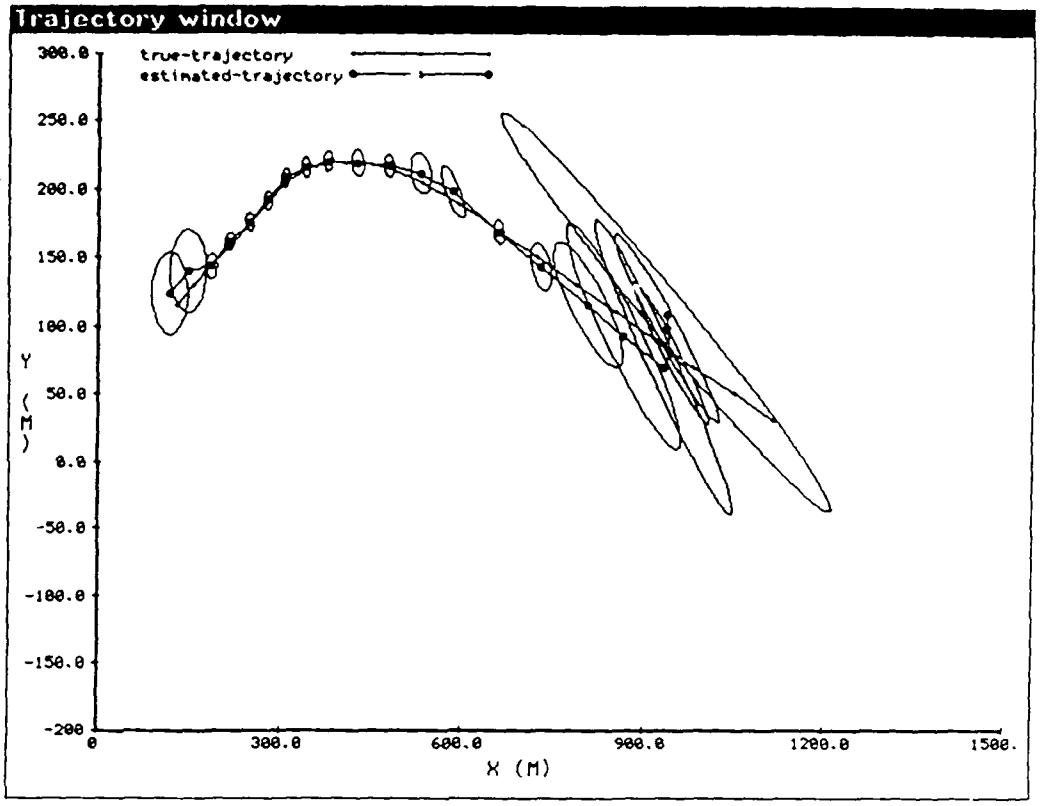


FIG. 4. Tracking results with sensor 2 only (one sample run).

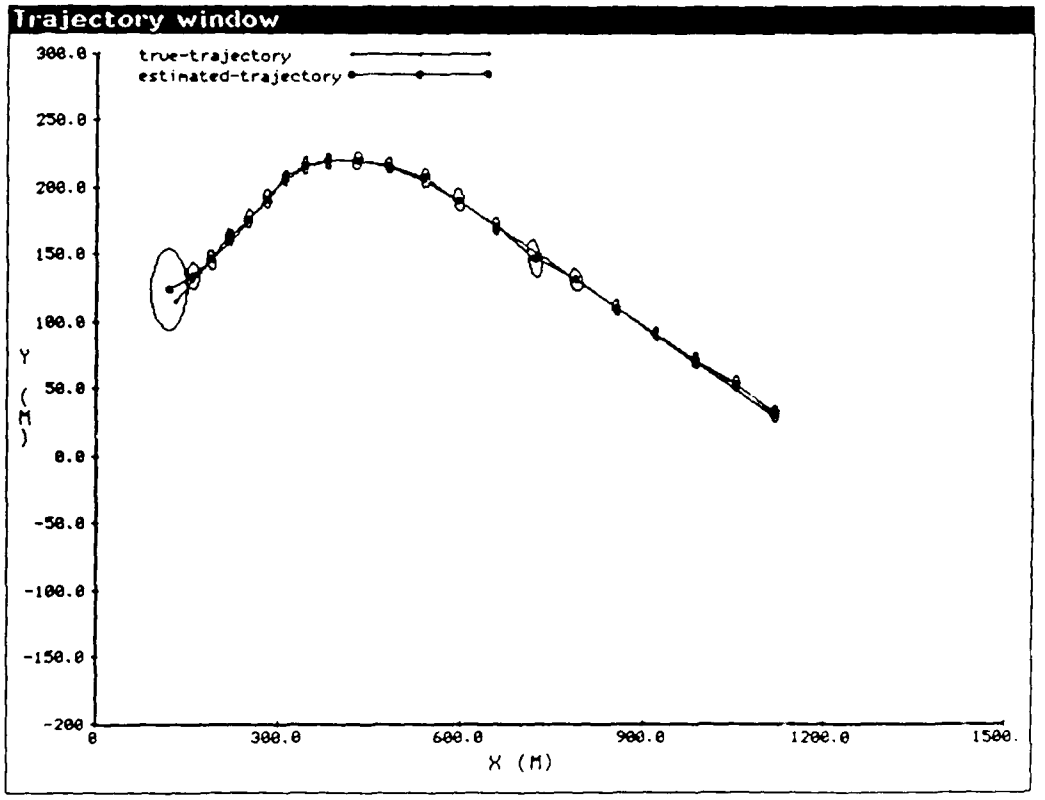


FIG. 5. Tracking results of distributed case (one sample run).

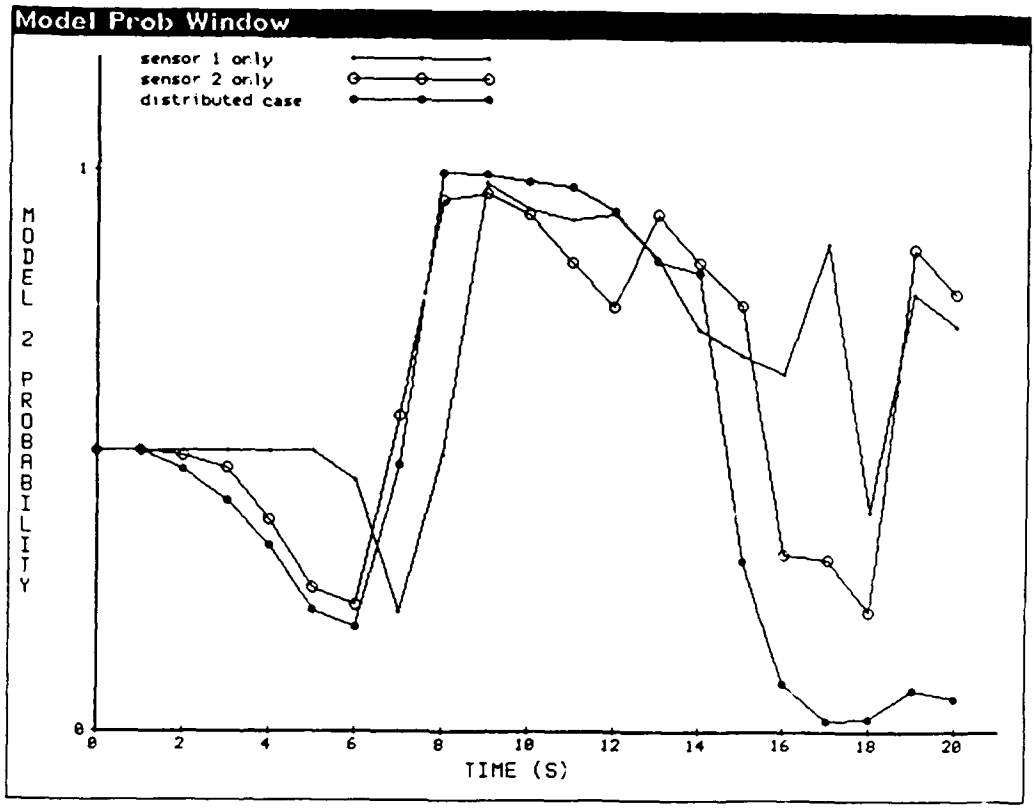


FIG. 6. Model 2 probability trajectories.

ation when the nodes communicate every scan. The simulation shows that the results of the distributed algorithm are the same as in the centralized algorithm, which confirms the theoretical equivalence.

## 6. CONCLUSION

A recursive estimation algorithm that accounts for the uncertainties of both measurement origins and system models in a distributed framework has been derived. The distributed estimation technique has been adopted together with the probabilistic data association (PDA) filter in conjunction with the interactive multiple model (IMM) scheme. The resulting algorithm can be applied to track a maneuvering target in a cluttered environment with distributed sensors. Simulation results show the expected performance of the algorithm. With full communication rate, the distributed case performs exactly the same as the centralized case, which confirms the theoretical equivalence, but has the advantages of increased reliability.

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TABLE 1. SIMULATION RESULTS (50 RUNS)

| | Decentralized case | | | |
|---|---|---|---|---|
| | Node 1 | Node 2 | Centralized case | Distributed case (full rate communication) |
| Number of successful tracks | 27 | 30 | 43 | 43 |
| r.m.s. Position error | 10.943 | 9.368 | 3.055 | 3.055 |

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Substituting these into (3.10), and using (E-2), we obtain

$$\lim_{t\to\infty} e_1(t)=0. \tag{3.13}$$

The estimation property (E-3), the uniform boundedness of $y(t)$ and $u(t)$, and (2.5) the definition of $t_j^*$, imply that

$$\lim_{t\to\infty} e(t)=0.$$

Substituting this into (3.11) and, again, using (E-2) we obtain

$$\lim_{t\to\infty} e_2(t)=0. \tag{3.14}$$

Since $E(z^{-1})$ is a stable polynomial, we can establish ii) by substituting (3.13) and (3.14) into (2.12). ▽▽▽

*Remark 3.1:* The multirate sampling estimation algorithm in general does not have the property that $e(t)/[1+\|\phi(t-1)\|^2]^{1/2}\in l_2$, which is required in the stability proof of conventional adaptive control algorithms. However, we still prove the stability using property (E-3) and the relation $|e(t_j^*)|\geq|e(t)|$ for $t_{j-1}\leq t<t_j$.

## IV. Conclusions

In this note, we have developed a multirate sampling adaptive control algorithm which allows a fast sampling rate of feedback control to be used even if the computation of parameter estimate and controller coefficient may take a relatively long period of time.

The key idea to achieve this is to record the plant input and output prior to the currently obtained estimate and use them to compute the coming estimate and controller coefficients. Thus, the computation is not dependent upon the inputs and outputs appearing during the updating process. The closed-loop system is shown to be stable.

*Remark 4.1:*

i) One may further extend the algorithm to consider $t_j-t_{j-1}>n+m+d=\bar{n}$. In this case, a relation

$$|e(t_{j-1}+\bar{n}+k)|\leq C_1\max_{t_{j-1}\leq t<t_{j-1}+\bar{n}}|e(t)|+C_2$$

($k<\infty$, $C_1<\infty$, $C_2<\infty$), can be used, and the algorithm only needs to compute $e(t)$ for $t_{j-1}\leq t<t_{j-1}+\bar{n}$ but not for every $t$ in $t_{j-1}\leq t<t_j$.

ii) Instead of the ARMA model, one can use $\delta$-model [8] in the algorithm, which retains the key features of the continuous-time model and allows a wide bandwidth MRAC system to be achieved.

iii) The multirate sampling adaptive control is presented for an indirect MRAC system. However, the method covers a wide class of direct and indirect adaptive control algorithms of certainty equivalence type such as pole-assignment, LQ-optimal, etc.

iv) Various methods developed for improving adaptive control system performance are applicable to the presented multirate sampling adaptive algorithm. These methods include: a) various modifications of parameter estimator for improving convergence rate; b) noise and disturbance filtering techniques; c) robustness techniques with respect to disturbances and unmodeled dynamics, such as deadzone, normalization, etc.; d) internal model principle for deterministic disturbance rejection, etc.

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# An Adaptive Dual Controller for a MIMO-ARMA System

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*Abstract*—**An adaptive dual controller is presented here for a multiinput multioutput ARMA system. The plant has constant but unknown parameters. The cautious controller with a one-step horizon and a new dual controller with a two-step horizon are examined. In many instances, the myopic cautious controller is seen to turn off and converges very slowly. The dual controller modifies the cautious control design by numerator and denominator correction terms which depend upon the sensitivity functions of the expected future cost and avoids the turn-off and slow convergence. Monte-Carlo comparisons based on parametric and nonparametric statistical analysis indicate the superiority of the dual controller over the cautious controller.**

## I. Introduction

Multiinput multioutput systems with unknown parameters are encountered in many practical situations, and their control poses a great challenge to the stochastic control theory. It is not possible to obtain an optimal solution for such systems because of the dimensionality involved in the stochastic dynamic programming [6]. In such situations, emphasis is on obtaining a suboptimal solution that incorporates the intrinsic properties of the optimal solution. For stochastic systems, the control has in general a dual effect [2], [11]: it affects the system's state as well as the future state and/or parameter uncertainty. Thus, the dual controller offers significant improvement potential for the control of uncertain linear plants. In multistage problems it "probes" the system to enhance real-time identification of the system's parameters in order to increase the accuracy of the subsequent control decisions and regulates the system at the same time [4], [9].

Two classes of dual controllers exist presently [14]. In the first class [10], [12], [18], the control minimizes a one-step ahead criterion augmented by a second term which penalizes for poor identification. This approach is simple but often requires tuning of some parameters. The second class (developed for SISO systems in [3], [16], [17]) used the stochastic dynamic programming equation and expands the future cost about a nominal trajectory. Using first- and second-order Taylor series expansions of the expected future cost about a nominal trajectory, dual controllers for MIMO static systems are developed in [5] and [14]. A second-order Taylor series expansion of the future expected cost is performed about a nominal trajectory and a dual controller based on a two-step horizon is developed in this note for a MIMO dynamic (ARMA) model. The cautious [14], [16], [18] and the new dual controller are applied to a MIMO-ARMA system. Monte Carlo simulations, based on parametric and nonparametric statistical analysis, indicate that the dual

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controller prevents the turn-off phenomenon and slow convergence prevalent with a cautious solution.

Section II gives the problem formulation. The approximate dual controller with a two-step horizon for the MIMO system is derived in Section III. The control solution is obtained by approximating the solution of the stochastic dynamic programming equation. A second-order Taylor series expansion of the expected future cost is performed about a nominal trajectory and this leads to a dual control solution in a closed form. Following the derivations of the controller, a summary of the algorithm is given. Section IV describes the simulation of the plant and compares the performances of the cautious and the dual solutions. Section V concludes the note.

## II. Problem Formulation

The MIMO system to be controlled is described by

$$y(k) = -Ay(k-1) + Bu(k-1) + e(k) \tag{1}$$

where

$$E[e(k)] = 0;\ E[e(k)\ e'(j)] = W\delta_{kj}. \tag{2}$$

Here $y(k)$ is the output of the plant, $u(k)$ is the input to the plant, and $e(k)$ is the measurement noise.

The parameter matrices $A$ and $B$ are unknown. This model describes some industrial processes like an ore crushing plant, or a heat exchanger [1]. The unknown elements of $A$ and $B$ comprise the parameter vector $\theta(k)$ whose estimate at time $k$ is $\hat{\theta}(k)$ with covariance matrix $P(k)$. The parameter vector is designated as

$$\theta(k) \triangleq [a_1' | b_1' | a_2' | b_2' | \cdots | a_n' | b_n']' \tag{3}$$

where $n$ is the dimension of the output vector $y(k)$ and $a_i'$, $b_i'$ are the $i$th row of the matrices $A$ and $B$, respectively. Assuming the parameters are time-invariant, we have

$$\theta(k+1) = \theta(k). \tag{4}$$

A measurement matrix $H(k)$ is defined as

$$H(k) \triangleq \text{diag}\,[-y'(k) | u'(k),\ -y'(k) | u'(k),\ \cdots] \tag{5}$$

where $H(k)$ has $n$ rows, and $y'(k)$, $u'(k)$ are the measurement and control vectors transposed.

With these definitions, the measurement model is

$$y(k) = H(k-1)\theta(k-1) + e(k). \tag{6}$$

The performance criterion to be minimized is $J(0)$, i.e., the conditional expected value of the cost $C(0)$ from step 0 to $N$, denoted by

$$J(0) = E\{C(0) | I^k\} = E\left[\sum_{k=0}^{N-1} \{y(k+1) - y_r\}' Q(k) \{y(k+1) - y_r\} | I^k\right] \tag{7}$$

where $Q(k)$ is the diagonal weighting matrix, $I^k$ is the cumulated information at time $k$, and $y_r$ is the desired output.

## III. Dual Control with a Two-Step Horizon

First the controller is derived and then a summary of the algorithm is provided.

A dual control solution with a two-step horizon is obtained by minimizing (2.7) with respect to the control $u(0)$ for the multidimensional plant (2.1)–(2.4). This is obtained by solving the general equation of stochastic dynamic programming [3], [7], [8]

$$J^*(k) = \min_{u(k)} E\{C(k) + J^*(k+1) | I^k\} \qquad k = N-1, \cdots, 1, 0 \tag{1}$$

where $J^*(k)$ is the optimal expected *cost to go* from $k$ to $N$, $C(k)$ is the cost to go from $k$ to $N$, and $I^k$ is the cumulated information at time $k$ when the control $u(k)$ is to be applied. The information $I^k$ is the set of all past controls until time $k-1$ and outputs until time $k$.

Thus, for a two-step horizon we have

$$J^*_{k,k+2} = \min_{u(k)} E\{C(k) + J^*_{k+1,k+2} | I^k\} = \min_{u(k)} E[\{y(k+1) - y_r\}' Q(k) \{y(k+1) - y_r\} + J^*_{k+1,k+2} | I^k] \tag{2}$$

where $J^*_{k+1,k+2}$ is the optimal expected cost at the last step with one-step horizon and is obtained by minimization of $J_{k+1,k+2}$, and $J_{k+1,k+2}$ is the cost to go from $k+1$ to $k+2$.

The cautious control at $k+1$ with one-step horizon is given by

$$u(k+1) = [E\{B'Q(k+1)B | I^{k+1}\}]^{-1} \cdot E[B'Q(k+1)\{Ay(k+1) + y_r\} | I^{k+1}]. \tag{3}$$

The cost from step $k+1$ to $k+2$ is

$$J_{k+1,k+2} = \text{tr}\, Q(k+1)W + E[\{Ay(k+1) + y_r\}' Q(k+1) \{Ay(k+1) + y_r\} + u'(k+1)B'Q(k+1)Bu(k+1) - 2\{Ay(k+1) + y_r\}' \cdot Q(k+1)Bu(k+1) | I^{k+1}] \tag{4}$$

and inserting (3) into (4) the optimal cost at the last step is

$$J^*_{k+1,k+2} = \text{tr}\, Q(k+1)W + E[\{Ay(k+1) + y_r\}' Q(k+1) \{Ay(k+1) + y_r\} | I^{k+1}] - E\{\{Ay(k+1) + y_r\}' Q(k+1)B | I^{k+1}] \cdot [E\{B'Q(k+1)B | I^{k+1}\}]^{-1} \cdot E[B'Q(k+1)\{Ay(k+1) + y_r\} | I^{k+1}] \tag{5}$$

where $E\{\cdot | I^{k+1}\}$ is the conditional expectation given the available information $I^{k+1}$.

The unknown parameters will be chosen from the Gaussian family and thus their estimate $\hat{\theta}(k+1)$ and associated error covariance $P(k+1)$ are the sufficient statistic. The parameter vector estimate $\hat{\theta}(k+1)$ and the associated covariance matrix $P(k+1)$ are obtained from a Kalman filter according to

$$K(k+1) = P(k)H'(k)[H(k)P(k)H'(k) + W]^{-1} \tag{6}$$

$$\hat{\theta}(k+1) = \hat{\theta}(k) + K(k+1)[y(k+1) - H(k)\hat{\theta}(k)] = \hat{\theta}(k) + K(k+1)\nu(k+1) \tag{7}$$

$$P(k+1) = P(k) - P(k)H'(k)[H(k)P(k)H'(k) + W]^{-1}H(k)P(k). \tag{8}$$

Here $\nu(k+1)$ is the innovation of the process.

From (5) it is clear that $J^*_{k+1,k+2}$ is a nonlinear function of the estimated parameter vector $\hat{\theta}(k+1)$ and covariance $P(k+1)$. But the estimated vector $\hat{\theta}(k+1)$ and the covariance $P(k+1)$ are not known until the control $u(k)$ is applied.

A control $u(k)$ with a two-step horizon can be obtained from (2) if a second-order Taylor series expansion of $J^*_{k+1,k+2}$ is performed about a suitable nominal trajectory. Here the nominal trajectory is defined by

1) a nominal parameter estimate $\bar{\theta}(k+1) = \hat{\theta}(k)$
2) a nominal control $\bar{u}(k)$
3) a nominal covariance $\bar{P}(k+1)$ obtained by using $\bar{u}(k)$
4) a nominal measurement $\bar{y}(k+1)$ obtained by using $\bar{u}(k)$ and $\hat{\theta}(k)$, i.e., $\bar{y}(k+1) = \bar{H}(k)\hat{\theta}(k)$.

Expansion of (5) about this nominal trajectory results in

$$J^*_{k+1,k+2} = J_1 + J'_y(k+1)[y(k+1)-\bar{y}(k+1)] + \frac{1}{2}[y(k+1)-\bar{y}(k+1)]' J_{yy}(k+1)[y(k+1)-\bar{y}(k+1)] + J'_\theta(k+1)[\hat{\theta}(k+1)-\hat{\theta}(k)] + \frac{1}{2}[\hat{\theta}(k+1)-\hat{\theta}(k)]' \cdot J_{\theta\theta}(k+1)[\hat{\theta}(k+1)-\hat{\theta}(k)] + \operatorname{tr}[J_P(k+1)\{P(k+1)-\bar{P}(k+1)\}] \quad (9)$$

where $J_1$ is the zeroth-order term and the cost sensitivities are

$$J_y(k+1) \triangleq \left[\frac{\partial J^*_{k+1,k+2}}{\partial y_i(k+1)}\right] \quad (10)$$

$$J_{yy}(k+1) \triangleq \left[\frac{\partial^2 J^*_{k+1,k+2}}{\partial y_i(k+1)\partial y_j(k+1)}\right] \quad (11)$$

$$J_\theta(k+1) \triangleq \left[\frac{\partial J^*_{k+1,k+2}}{\partial \hat{\theta}_i(k+1)}\right] \quad (12)$$

$$J_{\theta\theta}(k+1) \triangleq \left[\frac{\partial^2 J^*_{k+1,k+2}}{\partial \hat{\theta}_i(k+1)\partial \hat{\theta}_j(k+1)}\right] \quad (13)$$

$$J_P(k+1) \triangleq \left[\frac{\partial J^*_{k+1,k+2}}{\partial P^{ij}(k+1)}\right]. \quad (14)$$

The above sensitivities are evaluated at $\hat{\theta}(k)$, $\bar{P}(k+1)$, and $\bar{y}(k+1)$; and $P^{ij}(k+1)$ is the $ij$th element of the covariance matrix associated with the parameter estimates $\hat{\theta}_i(k+1)$ and $\hat{\theta}_j(k+1)$.

Under the Gaussian assumption for the zero mean noise

$$y(k+1)-\bar{y}(k+1) \sim \mathfrak{N}[\mu, V] \quad (15)$$

where the conditional mean is

$$\mu = E\{H(k)\theta(k)+e(k+1)-\bar{H}(k)\hat{\theta}(k)|I^k\} = [H(k)-\bar{H}(k)]\hat{\theta}(k) \quad (16)$$

and the conditional covariance is

$$V = E[\{y(k+1)-\bar{y}(k+1)-\mu\}\{y(k+1)-\bar{y}(k+1)-\mu\}'|I^k] = H(k)P(k)H'(k)+W. \quad (17)$$

With the choice of the nominal path as defined earlier and using (6), (16), and (17), the conditional expected value of (9) is

$$E\{J^*_{k+1,k+2}|I^k\} = J_1 + J'_y(k+1)[H(k)-\bar{H}(k)]\hat{\theta}(k) + \frac{1}{2}\mu' J_{yy}(k+1)\mu + \frac{1}{2}\operatorname{tr}[J_{yy}(k+1)V] + \frac{1}{2}\operatorname{tr}[J_{\theta\theta}(k+1)\{P(k)-P(k+1)\}] + \operatorname{tr}[J_P(k+1)\{P(k+1)-\bar{P}(k+1)\}]. \quad (18)$$

The above expected future cost (18) is a function of the nominal parameters multiplied by appropriate sensitivity functions $J_y(k+1)$, $J_{yy}(k+1)$, $J_{\theta\theta}(k+1)$, and $J_P(k+1)$. These sensitivities introduce the dual effect into (2) which is then used to yield $u(k)$. It must also be noted that the covariance $P(k+1)$ is nonlinear in $u(k)$ and is not yet known. Hence, a second-order expansion of $P(k+1)$ is proposed about a nominal control $\bar{u}(k)$ and a nominal covariance $\bar{P}(k+1)$ in order to obtain a (suboptimal) dual solution $u_D(k)$ in a *closed form* from (2).

This expansion is performed as follows:

$$P(k+1) \simeq \bar{P}(k+1) + \sum_{ij} e_i e_j' \left\{ P^{ij}_u(k+1)[u(k)-\bar{u}(k)] + \frac{1}{2}[u(k)-\bar{u}(k)]' P^{ij}_{uu}(k+1)[u(k)-\bar{u}(k)] \right\} \quad (19)$$

with the superscript here denoting the matrix element, $e_i$ the $i$th Cartesian basis vector, and

$$p^{ij}_u(k+1) \triangleq \frac{\partial P^{ij}(k+1)}{\partial u(k)}; \quad P^{ij}_{uu}(k+1) \triangleq \frac{\partial^2 P^{ij}(k+1)}{\partial u^2(k)} \qquad i, j = 1, \cdots, r \quad (20)$$

evaluated at $\bar{P}(k+1)$ and $\bar{u}(k)$ and $r$ the number of unknown parameters.

Now a (suboptimal) dual solution $u_D(k)$ with a two-step horizon can be obtained from (2) using (18)–(20) and is given in closed form by

$$u_D(k) = [E\{B'Q(k)B|I^k\}+F]^{-1}[E\{B'Q(k)(Ay(k)+y_r)\}I^k\}+f] \quad (21)$$

where the elements of the matrix $F$ and those of the vector $f$ are given by

$$F_{ij} = \frac{1}{2}\operatorname{tr}\left[\left\{J_P(k+1)-\frac{1}{2}J_{\theta\theta}(k+1)\right\}\frac{\partial^2 P(k+1)}{\partial u_i(k)\partial u_j(k)}\right] + \frac{1}{2}\operatorname{tr}\left[J_{yy}(k+1)\frac{\partial H(k)}{\partial u_i(k)}P(k)\left(\frac{\partial H(k)}{\partial u_j(k)}\right)'\right] + \frac{1}{2}\operatorname{tr}\left[J_{yy}(k+1)\left(\frac{\partial H(k)}{\partial u_i(k)}\hat{\theta}(k)\right)\left(\frac{\partial H(k)}{\partial u_j(k)}\hat{\theta}(k)\right)'\right] \qquad i, j = 1, \cdots, m \quad (22)$$

and

$$f_i = -\frac{1}{2}\left(\frac{\partial H(k)}{\partial u_i(k)}\hat{\theta}(k)\right)' J_y(k+1) - \frac{1}{2}\operatorname{tr}\left[\left\{J_P(k+1)-\frac{1}{2}J_{\theta\theta}(k+1)\right\}\frac{\partial P(k+1)}{\partial u_i(k)}\right] + \frac{1}{2}\sum_{j=1}^{m}\operatorname{tr}\left[\left\{J_P(k+1)-\frac{1}{2}J_{\theta\theta}(k+1)\right\}\frac{\partial^2 P(k+1)}{\partial u_i(k)\partial u_j(k)}\right]\bar{u}_j(k) + \frac{1}{2}\sum_{j=1}^{m}\operatorname{tr}\left[J_{yy}(k+1)\left(\frac{\partial H(k)}{\partial u_i(k)}\hat{\theta}(k)\right)\left(\frac{\partial H(k)}{\partial u_j(k)}\hat{\theta}(k)\right)'\right]\bar{u}_j(k) \quad (23)$$

and $m$ is the dimension of the control vector, $u_i$ is the $i$th element of the control vector.

It is clear from (21) that this approximate dual solution $u_D(k)$ is a modification of the cautious solution by the cost sensitivity terms. The cautious solution is (21) with $F = 0$ and $f = 0$. These account for the dual effect. The implementation of this second-order dual solution is performed by the method described below.

*Algorithm Summary:*

1) Compute the sensitivity functions $J_{\theta\theta}(k+1)$, $J_P(k+1)$, $J_y(k+1)$, $J_{yy}(k+1)$ for (18) with $\bar{\theta}(k+1) = \hat{\theta}(k)$ and the nominal values $\bar{u}(k)$, $\bar{P}(k+1)$, $\bar{y}(k+1)$ defining the nominal path.

2) Search on (2) with (18) [with the sensitivity functions computed above, starting with first nominal values $\bar{u}(k)$, $\bar{P}(k+1)$] over $u(k)$ to obtain an improved nominal for which $J^*_{k,k+2}$ is lower. This search is done by selecting a first coarse grid. A grid search is necessary to avoid locking in on a local minimum. Then another grid is chosen about the latter control over a narrower interval and from a second search $u^1(k)$ is obtained.

3) Using $u^1(k)$ compute the covariance sensitivities $P_u(k+1)$, $P_{uu}(k+1)$; together with the previously computed cost sensitivities $J_{\theta\theta}(k+1)$, $J_P(k+1)$, $J_{yy}(k+1)$, $J_y(k+1)$ obtain $F$, $f$ defined in (22), (23). Finally, the control to be applied, $u_D(k)$, is calculated from its explicit expression (21).

The iteration described in step 2) above is carried out to obtain better covariance sensitivities. The control $u_D(k)$ could have been obtained directly from (21) by skipping step 2) above; however, as indicated in [13] and [14], this results in unsatisfactory performance. With this iteration of step 2), the "improved" sensitivities yield good performance as shown in the next section.

## IV. Simulation Results

Performance is evaluated from 500 Monte Carlo runs for the following controllers:

1) heuristic certainty equivalence [3] (with a one-step horizon);
2) one-step ahead cautious controller; and
3) dual controller based upon sensitivity functions (with a two-step horizon) derived in Section III.

The plant equations for a two-input two-output system are

$$y_1(k+1) = -a_{11}y_1(k) - a_{12}y_2(k) + b_{11}u_1(k) + b_{12}u_2(k) + e_1(k+1) \quad (1)$$

$$y_2(k+1) = -a_{21}y_1(k) - a_{22}y_2(k) + b_{21}u_1(k) + b_{22}u_2(k) + e_2(k+1) \quad (2)$$

where

$$E\{e(k)e'(j)\} = W\delta_{kj} = \text{diag}\,(W_1, W_2);$$

$$W_1 = 7.52^2;\ W_2 = 43^2. \quad (3)$$

The true values of the parameters are

$$a_{11} = 0.8 \qquad b_{11} = -74.84$$
$$a_{12} = 0.1 \qquad b_{12} = -51.04$$
$$a_{21} = 0.2 \qquad b_{21} = \ \ 53.31$$
$$a_{22} = 0.75 \qquad b_{22} = -82.56. \quad (4)$$

Only the gain parameters ($B$ matrix) are considered unknown for testing the dual effect and their initial estimates were generated as $\mathcal{N}(b_{ij}, b_{ij}^2)$, $i, j = 1, 2$. This choice of system was motivated by the helicopter vibration study [13].

A large initial uncertainty is chosen in the parameter estimates in order to test the learning capabilities of the various adaptive algorithms. The cost weighting matrices are

$$Q(k) = \text{diag}\,(q_1, q_2); \qquad q_1 = 1.0,\ q_2 = 1.0. \quad (5)$$

The desired response is

$$y_r = [-18\ \ 80]'. \quad (6)$$

For the model chosen (1)–(6) the optimal control solution in order to reach a steady-state value of $y_r$ in (6) is

$$u_1^* = 1.0,\ u_2^* = -1.0. \quad (7)$$

In terms of the notation of (1) and (2)

$$\theta(k) \triangleq [a_{11}\ a_{12}\ b_{11}(k)\ b_{21}(k)\ a_{21}\ a_{22}\ b_{21}(k)\ b_{22}(k)]' \quad (8)$$

and

$$H(k) \triangleq \begin{bmatrix} -y_1(k) & -y_2(k) & u_1(k) & u_2(k) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -y_1(k) & -y_2(k) & u_1(k) & u_2(k) \end{bmatrix}. \quad (9)$$

The controllers are implemented with a *sliding horizon* for a total of 40 time steps. The evaluation criterion is

$$C_k = (y(k+1) - y_r)'\,Q(k)(y(k+1) - y_r). \quad (10)$$

### A. Analysis of the Monte Carlo Average Costs

Comparisons are made between the performances of the cautious and the dual algorithm on the system and a statistical significance analysis is done using the normal theory approach (i.e., it is assumed that the central limit theorem holds for the sample mean from a large number of runs) [14]. Tables I–IV contain the results of the simulation runs. Table I compares the average cost $\bar{C}_k$ over 500 Monte Carlo runs for the first 40 time steps for HCE, cautious and the dual algorithms, with a control limiter $|u_i| \le 2$, $i = 1, 2$.

Clearly it is seen that the cumulative average cost is the lowest for the dual controller. The HCE incurs an excessive penalty in time step 1) because of lack of caution. The cautious controller is overly cautious and exhibits slow convergence. However, the dual controller incurs less penalty in time step 1) than the HCE and makes a judicious choice of caution and probing to learn the parameters fast. Fig. 1 compares the performances of the three algorithms for 500 Monte Carlo runs. Both Table I and Fig. 1 demonstrate the *superior rate of adaptation of the dual algorithm*.

Table II provides a statistical significance test and shows the *improved performances* of the dual solution from time step 4) onwards with at least 98 percent confidence.

Table III indicates the percentage of runs where the cost exceeds 2000 for the two algorithms. This threshold of 2000 is selected from a sample distribution study of the cost at each time step. Table IV shows the percentile test [14], [15] comparing the cautious and the dual solution. They clearly indicate from time step 4) onwards the *light tailed nature* of the distribution of the cost yielded by the *new dual control algorithm*.

TABLE I
AVERAGE COSTS FOR THE THREE ALGORITHMS IN THE SIMULATION WITH A LIMITER ($|u_1| \le 2.0$, $|u_2| \le 2.0$) (500 MONTE CARLO RUNS). THE SUPERIOR RATE OF ADAPTATION OF THE DUAL ALGORITHM IS DEMONSTRATED HERE

| Time Step | HCE | | Cautious | | Dual | |
|---|---|---|---|---|---|---|
| $k$ | $\bar{C}_k$ | $\sum_{i=1}^{k} \bar{C}_i$ | $\bar{C}_k$ | $\sum_{i=1}^{k} \bar{C}_i$ | $\bar{C}_k$ | $\sum_{i=1}^{k} \bar{C}_i$ |
| 1 | 14851 | 14851 | 3623 | 3623 | 6944 | 6944 |
| 2 | 6241 | 21092 | 3961 | 7584 | 6722 | 13666 |
| 3 | 3578 | 24670 | 3246 | 10830 | 4230 | 17896 |
| 4 | 1616 | 26286 | 2836 | 13666 | 1866 | 19762 |
| 5 | 1354 | 27640 | 2505 | 16171 | 1492 | 21254 |
| 6 | 807 | 28447 | 2154 | 18325 | 953 | 22207 |
| 7 | 593 | 29040 | 1921 | 20246 | 700 | 22907 |
| 8 | 462 | 29502 | 1670 | 21916 | 582 | 23489 |
| 9 | 397 | 29899 | 1623 | 23539 | 535 | 24024 |
| 10 | 347 | 30246 | 1327 | 24866 | 385 | 24409 |
| . | . | . | . | . | . | . |
| . | . | . | . | . | . | . |
| 40 | 77 | 34444 | 281 | 43810 | 89 | 29178 |

TABLE II
STATISTICAL SIGNIFICANCE TEST FOR COMPARISONS OF THE CAUTIOUS AND THE DUAL ALGORITHM IN THE SIMULATION WITH A LIMITER ($|u_1| \le 2.0$, $|u_2| \le 2.0$) (500 MONTE CARLO RUNS)

| Time Step $k$ | Test Statistic $Z_k$ | Estimated Improvement $EI_k$ % |
|---|---|---|
| 1 | -8.1 | -91 |
| 2 | -5.3 | -69 |
| 3 | -2.2 | -30 |
| 4 | 3.5 | 34 |
| 5 | 3.3 | 40 |
| 6 | 6.0 | 56 |
| 7 | 6.3 | 64 |
| 8 | 6.5 | 65 |
| 9 | 6.5 | 67 |
| 10 | 5.7 | 71 |
| 11 | 6.3 | 76 |
| 12 | 5.6 | 70 |
| 13 | 5.9 | 82 |
| 14 | 5.2 | 62 |
| 15 | 5.5 | 79 |
| 16 | 4.9 | 70 |
| 17 | 4.5 | 78 |
| 18 | 4.4 | 74 |
| 19 | 4.4 | 76 |
| 20 | 4.3 | 76 |

### B. Individual Time History Runs

Analysis of the Monte Carlo average cost indicates the improvement offered by the dual solution; it provides no information about the cautious control's *turning-off phenomenon* [16], [18]. Hence, a careful investigation of the individual runs is required to examine these occurrences.

TABLE III
COMPARISON OF THE TAILS USING THE CAUTIOUS AND THE DUAL ALGORITHMS IN THE SIMULATION WITH A LIMITER ($|u_1| \leq 2.0$, $|u_2| \leq 2.0$) (500 MONTE CARLO RUNS)

| Time Step k | Percentage of runs which exceed 2000 | |
|---|---|---|
| | Cautious | Dual |
| 1 | 86 | 76 |
| 2 | 60 | 52 |
| 3 | 43 | 40 |
| 4 | 33 | 25 |
| 5 | 31 | 17 |
| 6 | 22 | 10 |
| 7 | 22 | 8 |
| 8 | 19 | 7 |
| 9 | 16 | 3 |
| 10 | 12 | 2 |
| 11 | 12 | 1.2 |
| 12 | 10 | 1.4 |
| 13 | 11 | 1.4 |
| 14 | 7 | 1 |
| 15 | 8 | 0.4 |
| 16 | 6 | 0.4 |
| 17 | 6 | 0.2 |
| 18 | 6 | 0.4 |
| 19 | 5 | 0.4 |
| 20 | 5 | 0.2 |

TABLE IV
PERCENTILE TEST FOR COMPARISONS OF THE CAUTIOUS AND THE DUAL ALGORITHMS IN THE SIMULATION WITH A LIMITER ($|u_1| \leq 2.0$, $|u_2| \leq 2.0$) (500 MONTE CARLO RUNS)

| Time Step k | $X^2$ test statistics at $K_{95}$ |
|---|---|
| 1 | -- |
| 2 | -- |
| 3 | -- |
| 4 | 10 |
| 5 | 19 |
| 6 | 23 |
| 7 | 32 |
| 8 | 35 |
| 9 | 57 |
| 10 | 37 |
| 11 | 40 |
| 12 | 40 |
| 13 | 40 |
| 14 | 16 |
| 15 | 32 |
| 16 | 11 |
| 17 | 16 |
| 18 | 16 |
| 19 | 18 |
| 20 | 25 |

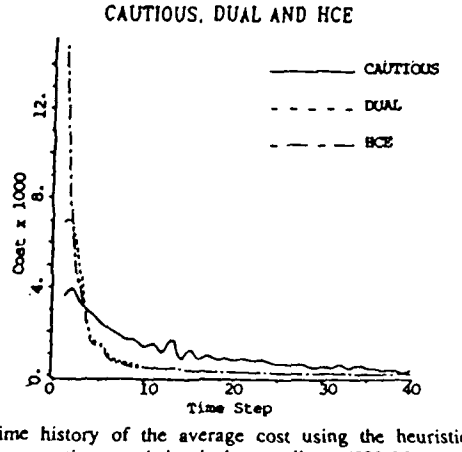


Fig. 1. Time history of the average cost using the heuristic certainty equivalence, cautious, and the dual controllers. (500 Monte Carlo runs; $|u_1| \leq 2.0$, $|u_2| \leq 2.0$.) The superior rate of adaptation of the dual algorithm is demonstrated here.

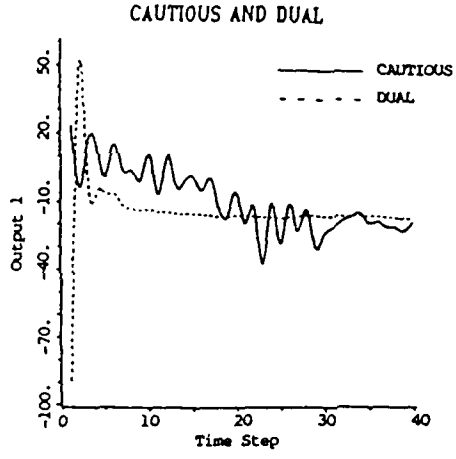


Fig. 2. Time history of output 1 using the cautious and the dual algorithms for run 90 (500 Monte Carlo runs; $|u_1| \leq 2.0$; $|u_2| \leq 2.0$).

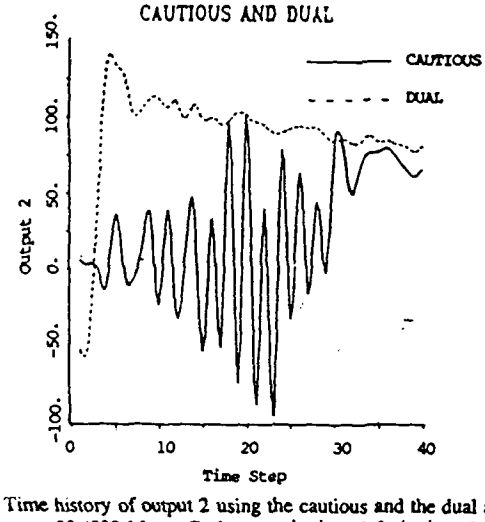


Fig. 3. Time history of output 2 using the cautious and the dual algorithms for run 90 (500 Monte Carlo runs; $|u_1| \leq 2.0$; $|u_2| \leq 2.0$).

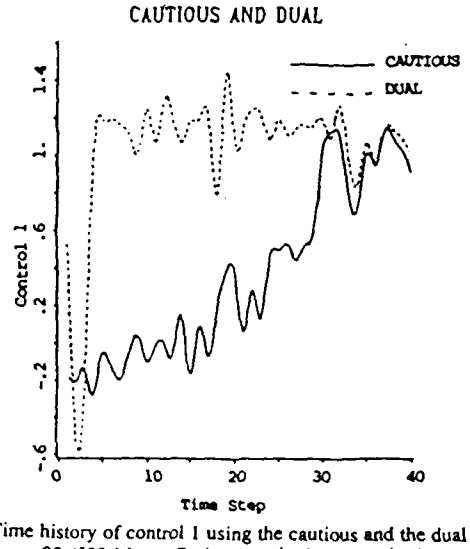


Fig. 4. Time history of control 1 using the cautious and the dual algorithms for run 90 (500 Monte Carlo runs; $|u_1| \leq 2.0$; $|u_2| \leq 2.0$).

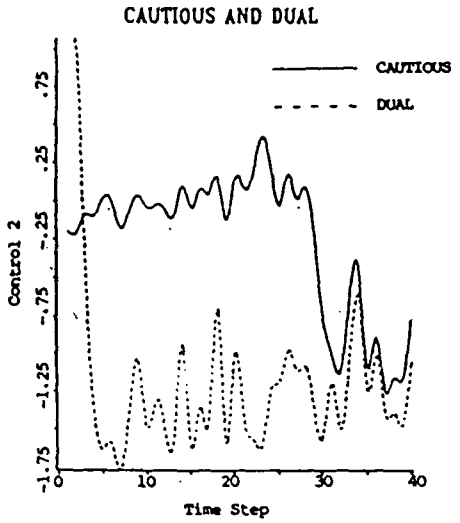


Fig. 5. Time history of control 2 using the cautious and the dual algorithms for run 90 (500 Monte Carlo runs; $|u_1| \leq 2.0$; $|u_2| \leq 2.0$).

The turn-off phenomenon is observed in many runs among the 500 Monte Carlo simulations while using the cautious controller; run 90 is a typical example of it. Both components are almost off between time steps 0 and 20 during which the dual controller already identified the parameters and reached the desired trajectory. Figs. 2–5 portray this result.

## V. Conclusions

A new adaptive dual control solution with a two-step sliding horizon has been developed for an ARMA–MIMO system. The control law is derived by solving the stochastic dynamic programming equation. This solution utilizes the dual effect by performing a second-order Taylor series expansion of the expected future cost and does not need any tuning for any of the runs in the example. It modifies the cautious solution by explicit numerator and denominator correction terms. The controller in its present form is the first of its kind *in a closed form* for a system with unknown parameters. The controller is tested on a *MIMO system* in a *systematic* Monte Carlo fashion. Conclusions are based on 500 Monte Carlo runs. Analysis of the simulation runs has shown that this new *dual control solution* applied to a *multiinput multioutput model improves over the cautious controller*. The key improvement is in the *avoiding of situations* like *turn-off and slow convergences*, typical of the cautious solution.

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# TIME-REVERSION OF A HYBRID STATE STOCHASTIC DIFFERENCE SYSTEM

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## ABSTRACT

The reversion in time of a stochastic difference equation in a hybrid space, with a Markovian solution, is presented. The reversion is obtained by a martingale approach, which previously led to reverse time forms for stochastic equations with Gauss-Markov or diffusion solutions. The reverse time equations follow from a particular non-canonical martingale decomposition, while the reverse time equations for Gauss-Markov and diffusion solutions followed from the canonical martingale decomposition. The need for this non-canonical decomposition stems from the hybrid state space situation. Moreover, the non-Gaussian discrete time situation leads to reverse time equations that incorporate a Bayesian estimation step.

## 1. INTRODUCTION

This paper adresses the problem of time-reversion of a hybrid state Markov process which is given as the solution of a stochastic difference equation. The desired result is a similar equation but running in reverse-time direction while having a solution that is respectively pathwise and in probability law equivalent to the solution of the forward equation.

The motivation to study this problem stems from two different kinds of application. The first is to approach the solution of a nonlinear smoothing problem by a merging of the estimates of two nonlinear filters: one filter matches the original model and is applied in the usual time direction while the other filter matches the time-reversed model and is applied in the reverse-time direction. The second application is the determination of a rate distortion theory lower bound for a discrete-time nonlinear filtering problem by the method of Galdos. This method is based on Bucy's representation formula and requires a Monte Carlo simulation in reverse-time direction of model matching trajectories, starting from a prespecified end point (Galdos, 1981; Washburn et al., 1985). For both of these two applications it is necessary to have a time-reversed difference equation for which the Markovian solutions are in probability law equivalent to the original solution.

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Our problem falls in the category of how to reverse a Markov process in time. The Markov property implies that the past and the future are independent under the condition that the present state is known (Wentzell, 1981). This invariance with respect to the time direction is the key property used in time-reversion studies. There are two types of studies that deal with this problem; a classical type and a systems-type. The classical type of study assumes that the transition measure or the generator of a Markov process is given and then tries to characterize the transition measure in reverse-time direction (Nagasawa, 1964; Kunita and Watanabe, 1966; Chung and Walsh, 1969; Azéma, 1973; Hasegawa, 1976; Dynkin, 1978; Williams, 1979).

The systems-type of study assumes that a stochastic equation with a Markovian solution is given for which it tries to characterize the time-reversed equation. The first time-reversed equations were obtained by orthogonality arguments, for the linear Gaussian situation (Ljung and Kailath, 1976; Lainiotis, 1976). For general diffusions, it has already been pointed out by Stratonovich (1960) how to obtain the reversed-time equations by actually following the classical approach: from a stochastic equation via the generator and the time-reversed generator back to time-reversed equations. A truly systems-type of study has been started by Verghese and Kailath (1979), by showing how for a linear Gaussian system a more direct martingale approach leads in a simpler way to time-reversed equations. Moreover, by this approach it was possible to obtain a reversed-time equation with a pathwise equivalent solution. Early elaborations of these ideas led, along different routes, to time-reversed equations with pathwise equivalent solutions (Anderson, 1982; Castanon, 1982; Pardoux, 1983). During subsequent studies, quite large classes of stochastic differential equations and their reversed-time equations have been identified (Elliott and Anderson, 1985; Pardoux, 1985; Elliott, 1986a, 1986b; Haussmann and Pardoux, 1986; Pardoux, 1986). Recently these results have been extended by using the Girsanov transformation of Brownian motion (Picard, 1986; Protter, 1987). Obviously, this Girsanov approach can not be applied to discontinuous or discrete-time processes.

To give an idea of why there is an additional problem in using a martingale approach to the reversion of an equation with a discontinuous solution, we give a brief outline of the approach. The martingale approach roughly consists of checking if the time-reversed driving noise

sequence can be decomposed in a suitable reverse-time martingale part and its complement and next, if such a decomposition exists (Jacod and Shiryaev, 1987; Jacod and Protter, 1988), selecting such a decomposition. The final step is to characterize both the martingale part and its complement. In contrast with a continuous process such a decomposition is not unique for a discontinuous process (see for example, Jacod and Shiryaev, 1987). This makes the selection of a suitable martingale decomposition far from trivial in the hybrid state space situation, because a less good choice yields unnecessarily complicated reverse-time equations. This complication is presently unsolved, neither in continuous-time nor in discrete-time. It will be solved in the sequel for quite general difference equations in a hybrid space. With that result we subsequently reverse the considered difference equation in time.

The paper is organized as follows. In section 2 we define the hybrid state stochastic difference equation that will be considered and shortly compare its time-reversion with the time-reversion of a linear Gaussian equation. In section 3 we specify the time-reversion requirements. Next, in sections 4 and 5 we consider, respectively, the pathwise time-reversion and the in probability law equivalent time-reversion. In section 6 we discuss the results obtained.

## 2. THE STOCHASTIC DIFFERENCE EQUATION CONSIDERED

The stochastic difference equation we consider in the sequel is the following system, on an appropriate stochastic basis and a discrete time interval $[0,T] \in \mathbb{N}\times[0,T]$, $T<\infty$,

$$x_{t+1} = a(\theta_{t+1},\theta_t,x_t,w_t), \tag{1.a}$$
$$\theta_{t+1} = b(\theta_t,v_t), \tag{1.b}$$
$$y_t = c(\theta_t,x_t,w_t,u_t), \tag{1.c}$$

where $(w_t)$, $(u_t)$ and $(v_t)$ are i.i.d. standard Gaussian sequences of dimension p, q and 1 respectively, the initial distribution of $(x_0,\theta_0)$ has the density mass function $p_{x_0,\theta_0}$, and $(w_t,v_t,u_t)$ is independent of $(x_0,\theta_0)$. Further $x_t$, $\theta_t$ and $y_t$ have respectively $\mathbb{R}^n$-, $\mathbb{M}$- and $\mathbb{R}^m$-valued realizations (with $\mathbb{M}$ a countable set), while a, b and c are measurable mappings of appropriate dimensions such that system (1) has a unique solution for each initial $(x_0,\theta_0)$ with $p_{x_0,\theta_0}(x_0,\theta_0)\neq 0$. The mappings a, b and c are time-invariant for notational simplicity only.

The second order dependence of (1.a) on $(\theta_t)$ is quite uncommon (Blom, 1985). Obviously, (1.a) reduces to the more common situation of first order dependence, only if $a(\theta_{t+1},\theta_t,x_t,v_t)$ is invariant w.r.t. either $\theta_t$ or $\theta_{t+1}$. The interpretation of (1.a) as an equation with a second order dependence on $(\theta_t)$ suggests the substitution of $\xi_{t+1}\triangleq(\theta_{t+1},\theta_t)$ in (1.a). On doing this (1.a) reduces to the more common equation, and it follows immediately that $(\xi_t)$ and $(\xi_t,x_t)$ are Markov processes. However, as the state space of $\xi_t$ is significantly larger than the state space of $\theta_t$, this is a rather brute force transformation of (1.a). A more elegant transformation of (1.a) to the more common equation consists of substituting (1.b) in (1.a), which yields an equation of the following form,

$$x_{t+1} = a'(\theta_t,x_t,w_t,v_t).$$

Instead of a state space expansion, there appears an additional noise term, $v_t$. From the latter representation, it follows immediately that the processes $(\theta_t,x_t)$ and $(\theta_t)$ are Markov processes. The latter transformation clearly shows that (1.a) is indeed more general than the more commonly studied equation with first order dependence of $(\theta_t)$. With the study of this more general equation, we also anticipate the time-reversion results obtained. In the sequel it will turn out that a reverse-time equation of (1.a) has, in general, a second order dependence on the time-reversed $(\theta_t)$, even when $a(\theta_{t+1},\theta_t,x_t,w_t)$ is $\theta_t$-invariant. In view of this, it is natural to study the above more general form.

In the sequel we consider the time-reversion of system (1) under the following assumptions:

A.1
$a(\theta,\eta,.,w)$ has an inverse $a^*:\mathbb{M}^2\times\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}^n$, such that for any $(\theta,\eta,w)\in\mathbb{M}^2\times\mathbb{R}^p$,

$$a^*(\theta,\eta,a(\theta,\eta,x,w),w)=x; \quad \text{all } x\in\mathbb{R}^n. \tag{2}$$

A.2
$b(.,v)$ has an inverse $b^*:\mathbb{M}\times\mathbb{R}\to\mathbb{M}$, such that for any $v\in\mathbb{R}$,

$$b^*(b(\theta,v),v)=\theta; \quad \text{all } \theta\in\mathbb{M}. \tag{3}$$

Assumptions A.1 and A.2 suggest to transform (1.a,b,c) to the following time-reversed model,

$$x_t=a^*(\theta_{t+1},\theta_t,x_{t+1},w_t),$$
$$\theta_t=b^*(\theta_{t+1},v_t),$$
$$y_t=c(\theta_t,x_t,w_t,u_t).$$

Because $(w_t,v_t)$ and the future (= reverse-time past), $\mathcal{F}_{t+1} = \sigma((y_s,x_s,\theta_s); s\in[t+1,T])$, are dependent, this is not the time-reversed system we should look for. Unfortunately, it is not clear how to continue from here. To develop some insight, we take a quick look at the time-reversion of a linear Gaussian system.

Linear Gaussian example
Consider the following linear Gaussian system

$$x_{t+1} = Ax_t + Bw_t.$$

Assumption A.1 implies that A is invertible, by which

$$x_t = A^{-1}(x_{t+1} - Bw_t).$$

Obviously $w_t$ and the future $\mathcal{F}_{t+1}$ are dependent, which requires a martingale decomposition of $w_t$. In this linear Gaussian case the canonical martingale decomposition is the appropriate one. It consists of decomposing $w_t$ in its reverse-time predictable part, $E(w_t|\mathcal{F}_{t+1})$, and its complement $w^*_t$;

$$w_t = E(w_t|\mathcal{F}_{t+1}) + w^*_t.$$

The problem is now to write the predictable part as a function of $x_{t+1}$ (if possible) and to

characterize the covariance of $w^*_t$. As pointed out by Verghese and Kailath (1979) it follows readily from orthogonality arguments that

$$E(w_t|\mathcal{Y}_{t+1}) = E(w_t|x_{t+1}),$$

while the fundamental formula for LLSE estimation yields

$$E(w_t|x_{t+1}) = B^T R^{-1}(t+1)x_{t+1},$$
$$Cov(w^*_t) = I - B^T R^{-1}(t+1)B,$$

where $R(t+1)$ is the covariance of $x_{t+1}$. By a straightforward substitution of these results we obtain

$$x_t = A^{-1}\{x_{t+1} - B\,B^T R^{-1}(t+1)x_{t+1} - Bw^*_t\},$$

which yields the desired reverse-time system:

$$\bar{x}_t = A^{-1}\{\bar{x}_{t+1} - B\,B^T R^{-1}(t+1)\bar{x}_{t+1} - B\bar{w}_t\}.$$

The orthogonality arguments and the LLSE estimation step, used in the above procedure, prevent a straightforward extension of that procedure to equation (1). In the sequel we replace the orthogonality arguments and the LLSE estimation step respectively by Markov duality arguments and a Bayesian estimation step. Besides this, we have to select an appropriate martingale decomposition. Following the linear Gaussian case, the canonical martingale decomposition seems a good candidate:

$$(w_t,v_t) = (w_t^*,v_t^*) + E((w_t,v_t)|\mathcal{Y}_{t+1}).$$

Unfortunately, this decomposition leads to very complicated elaborations of the Bayesian estimation step. To avoid these complications, we use in this paper the following decomposition:

$$(w_t^*,v_t^*) = (w_t,v_t) - (\hat{w}_t,\hat{v}_t),$$

with: $\hat{v}_t \triangleq E(v_t|\mathcal{Y}_{t+1})$ and

$$\hat{w}_t \triangleq E(w_t|\mathcal{Y}_{t+1},v_t).$$

The main step, that must be carried out, is to prove that the latter is a martingale decomposition, and to elaborate on the Bayesian estimation step. For the presentation of these results a constructive approach is taken, starting with a precise description of the time-reversion objectives.

## 3. TIME-REVERSION OBJECTIVES

We want to obtain a time-reversed version of system (1), such that its solution, $(\bar{y}_t,\bar{x}_t,\bar{\vartheta}_t)$, is in some sense equivalent to $(y_t,x_t,\vartheta_t)$. To make this objective explicit it needs both a specification of what we mean by a time-reversion of (1), and a specification of the desired sense of process equivalence.

By a reverse-time system we mean a stochastic difference equation which starts at time T and runs in negative time direction on the interval [0,T]. We require from a time-reversion of system (1) that it does not change the state space and that the solution of the resulting reverse-time system represents the process $(\bar{y}_t,\bar{x}_t,\bar{\vartheta}_t)$. More specificly, $(\bar{y}_t,\bar{x}_t,\bar{\vartheta}_t)$ must be the solution of the following system of stochastic difference equations, all $t\in[0,T-1]$:

$$\bar{x}_t = \bar{A}(t,\bar{\vartheta}_{t+1},\bar{\vartheta}_t,\bar{x}_{t+1},\vartheta_t), \qquad (4.a)$$

$$\bar{\vartheta}_t = \bar{B}(t,\bar{\vartheta}_{t+1},\bar{x}_{t+1},\vartheta_t), \qquad (4.b)$$

$$\bar{y}_t = \bar{C}(t,\bar{\vartheta}_{t+1},\bar{\vartheta}_t,\bar{x}_{t+1},\bar{x}_t,\vartheta_t,u_t), \qquad (4.c)$$

where $\bar{A}$, $\bar{B}$ and $\bar{C}$ are deterministic mappings of appropriate dimensions and $(\vartheta_t,\vartheta_t)$ is a noise sequence to be specified. For a better understanding of (4) notice that the substitutions of (4.a) in (4.c) and of (4.b) in (4.a,c) transform (4) to a reverse-time system of the more common form:

$$\bar{x}_t = \bar{A}(t,\bar{\vartheta}_{t+1},\bar{x}_{t+1},\vartheta_t,\vartheta_t),$$

$$\bar{\vartheta}_t = \bar{B}(t,\bar{\vartheta}_{t+1},\bar{x}_{t+1},\vartheta_t),$$

$$\bar{y}_t = \bar{C}(t,\bar{\vartheta}_{t+1},\bar{x}_{t+1},\vartheta_t,\vartheta_t,u_t); \quad \text{all } t\in[0,T-1].$$

To be a useful reverse-time system, $(\vartheta_t,\vartheta_t)$ should, as much as possible, be independent of the future (= reversed-time past) information field

$$\bar{\mathcal{Y}}_{t+1} \triangleq \sigma((\bar{y}_s,\bar{x}_s,\bar{\vartheta}_s,\vartheta_s,u_s);\ s\in[t+1,T]).$$

A minimal requirement is then, that the conditional expectation of $(\vartheta_t,\vartheta_t)$, given $\bar{\mathcal{Y}}_{t+1}$, should be zero. Because $\bar{\mathcal{Y}}_t$ is a decreasing sequence of sigma algebras, the latter can most easily be put in martingale language (see Elliott, 1982; Kumar and Varaiya, 1986; and the definitions below):

$(\vartheta_t,\vartheta_t)$ in (4) should be a <u>reverse-time Martingale Difference sequence w.r.t. $\bar{\mathcal{Y}}_t$</u>.

<u>1 Definition</u>
Assume $(\mathcal{Y}_t;\ t\in[0,T])$ is an <u>increasing</u> sequence of information fields, i.e. $\mathcal{Y}_{s-1}\subset\mathcal{Y}_s$; any $s\in[1,T]$. A random sequence $(\xi_t)$ is said to be a Martingale Difference sequence w.r.t. $\mathcal{Y}_t$ iff for all $t\in[0,T]$,

(i) $\xi_t$ is $\mathcal{Y}_t$-measurable,
(ii) $E(|\xi_t|)<\infty$,
(iii) $E(\xi_t|\mathcal{Y}_s)=0$ a.s.; for all $s\in[0,t-1]$.

<u>2 Definition</u>
Assume $(\mathcal{Y}_t;\ t\in[0,T])$ is a <u>decreasing</u> sequence of information fields, i.e. $\mathcal{Y}_s\subset\mathcal{Y}_{s-1}$; any $s\in[1,T]$. A random sequence $(\xi_t)$ is said to be a <u>reverse-time Martingale Difference sequence w.r.t. $\bar{\mathcal{Y}}_t$</u> iff for all $t\in[0,T]$,

(i) $\xi_t$ is $\mathcal{Y}_t$-measurable,
(ii) $E(|\xi_t|)<\infty$,
(iii) $E(\xi_t|\mathcal{Y}_s)=0$ a.s. ; for all $s\in[t+1,T]$.

Having specified the desired type of reverse-time system, the next step is to specify the types of equivalence of solutions of systems (1) and (4), in which we are interested. For stochastic processes several useful types of equivalence have been defined and named in the past. We restrict ourselves to the two most important types of equivalence and their unambiguous names (Elliott, 1982; Jacod and Shiryaev, 1987):

- indistinguishable,
- equivalent in law.

Definitions are given below.

3 Definition

Two processes $(\xi_t)$ and $(\tilde{\xi}_t)$, $t\in[0,T]$, are said to be indistinguishable if they are defined on the same probability space $(\Omega,\mathcal{F},P)$ and

$$P(\ \xi_t = \tilde{\xi}_t\ ,\ \text{all } t\in[0,T]\ ) = 1. \tag{5}$$

4 Definition

Two processes $(\xi_t)$ and $(\tilde{\xi}_t)$, $t\in[0,T]$, are said to be equivalent in law, if they have the same state space, $\mathbb{E}$, and for all $0\le t_1<t_2<\ldots<t_k\le T$,

$$P((\xi_{t_1},\ldots,\xi_{t_k})\in dX) = P((\tilde{\xi}_{t_1},\ldots,\tilde{\xi}_{t_k})\in dX)\ , \tag{6}$$

for any $k$ and all measurable $dX\subset\mathbb{E}^k$.

For discrete-time processes (5) is satisfied if and only if, for all $t\in[0,T]$, $\xi_t=\tilde{\xi}_t$ almost surely. Our objective in the sequel is to obtain time-reversed systems of type (4), with solutions that are respectively indistinguishable and equivalent in law w.r.t. the solution of (1).

## 4 INDISTINGUISHABLE TIME-REVERSION

In this section we derive a type (4) version of system (1), such that their solutions,

$(\bar{y}_t,\bar{x}_t,\bar{\theta}_t)$ and $(y_t,x_t,\theta_t)$, are indistinguishable, and illustrate these results for a jump-linear example.
The first step of our derivation consists of a substitution of (2) and (3) in (1), to arrive at the in section 2 discussed time-reversed system,

$$x_t = a^*(\theta_{t+1},\theta_t,x_{t+1},w_t), \tag{7.a}$$
$$\theta_t = b^*(\theta_{t+1},v_t), \tag{7.b}$$
$$y_t = c(\theta_t,x_t,w_t,u_t). \tag{7.c}$$

Although (7) and (4) look similar, one requirement is not met: the driving noise in (7) is not a reverse-time Martingale Difference sequence w.r.t. the future information field

$$\mathcal{F}_t \triangleq \sigma\{(y_s,x_s,\theta_s,w_s,v_s,u_s);\ s\in[t,T]\}. \tag{8}$$

Therefore our next step is to introduce a particular reverse-time Martingale Difference sequence, $(w_t^*,v_t^*)$, as follows,

$$(w_t^*,v_t^*) = (w_t,v_t) - (\hat{w}_t,\hat{v}_t)\ , \tag{9.a}$$

with

$$\hat{v}_t \triangleq E(v_t|\mathcal{F}_{t+1}), \tag{9.b}$$

$$\hat{w}_t \triangleq E(w_t|\mathcal{F}_{t+1},v_t);\qquad \text{all } t\in[0,T-1]. \tag{9.c}$$

and $(w_T^*,v_T^*)=0$.

Notice that the definition of $\hat{w}_t$ differs significantly from the reverse-time predictable process $E(w_t|\mathcal{F}_{t+1})$. As such the decomposition in (9) is not the unique canonical decomposition (see Jacod and Shiryaev, 1987). The introduction of this non-canonical decomposition is a crucial step necessary for obtaining the time-reversion of hybrid state system (1).

In the sequel we verify that $(w_t^*,v_t^*)$ is indeed a reverse-time Martingale Difference sequence w.r.t. $\mathcal{F}_t$, and thus also w.r.t. $\mathcal{F}_t^* \triangleq \mathcal{F}_t \vee \sigma\{(w_s^*,v_s^*);\ s\in[t,T]\}$. Moreover we show that, due to the duality of the Markov property, $(\hat{w}_t,\hat{v}_t)$ is conditionally independent of $\mathcal{F}_{t+2}$ given $(x_{t+1},\theta_{t+1})$.

5 Theorem

Assume $(w_t,v_t)$, $(\hat{w}_t,\hat{v}_t)$ and $(w_t^*,v_t^*)$ satisfy (1) and (9). Then $(w_t^*,v_t^*)$ is a reverse-time Martingale difference sequence w.r.t. $\mathcal{F}_t^*$, while $\hat{w}_t$ and $\hat{v}_t$ satisfy:

$$\hat{w}_t = E(w_t|\theta_{t+1},\theta_t,x_{t+1}), \tag{10.a}$$
$$\hat{v}_t = E(v_t|\theta_{t+1},x_{t+1}),\quad \text{all } t\in[0,T-1]. \tag{10.b}$$

Proof: See Blom and Bar-Shalom (1989).

Theorem 5 implies that $\hat{w}_t$ and $\hat{v}_t$ can be written as

$$\hat{w}_t = f(t,\theta_{t+1},\theta_t,x_{t+1}), \tag{11.a}$$
$$\hat{v}_t = g(t,\theta_{t+1},x_{t+1}). \tag{11.b}$$

Substitution of (9.a) and (11.a,b) in (7.a,b,c) yields

$$x_t = \tilde{a}(t,\theta_{t+1},\theta_t,x_{t+1},w^*_t), \tag{12.a}$$
$$\theta_t = \tilde{b}(t,\theta_{t+1},x_{t+1},v^*_t), \tag{12.b}$$
$$y_t = \tilde{c}(t,\theta_{t+1},\theta_t,x_{t+1},x_t,w^*_t,u_t), \tag{12.c}$$

with,

$$\tilde{a}(t,\theta,\eta,x,w^*) = a^*(\theta,\eta,x,w^*+f(t,\theta,\eta,x)), \tag{13.a}$$
$$\tilde{b}(t,\theta,x,v^*) = b^*(\theta,v^*+g(t,\theta,x)), \tag{13.b}$$
$$\tilde{c}(t,\theta,\eta,x,z,w^*,u) = c(\eta,z,w^*+f(t,\theta,\eta,x),u). \tag{13.c}$$

The above result is summarized by the following corollary.

6 Corollary

Under assumptions A.1 and A.2, the solution $(\bar{y}_t,\bar{x}_t,\bar{\theta}_t)$ of the reverse-time system (4) is indistinguishable from the solution $(y_t,x_t,\theta_t)$ of system (1) if

(I) $(\bar{y}_T,\bar{x}_T,\bar{\theta}_T) = (y_T,x_T,\theta_T)$ a.s.,

(II) $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ satisfy (13.a,b,c),

(III) $(\bar{w}_t,\bar{v}_t) = (w_t^*,v_t^*)$ a.s. ; all $t\in[0,T-1]$, with $w^*_t$ and $v^*_t$ satisfying (9.a) and (10).

Jump-linear example

To illustrate the results obtained so far, let us consider the particular situation of a linear system with first order Markovian switching coefficients and observation noise independent of the system driving noise. Both $a(\theta,\eta,x,w)$ and $c(\eta,x,w,u)$ are then linear in $(x,w)$, while the first is $\eta$-invariant and the second is $w$-invariant, by which system (1) simplifies to,

$$x_{t+1} = A(\theta_{t+1})x_t + B(\theta_{t+1})w_t,$$
$$\theta_{t+1} = b(\theta_t,v_t),$$
$$y_t = G(\theta_t)x_t + H(\theta_t)u_t.$$

Then from Corollary 6 we readily find the indistinguishable time-reversed system,

$$x_t = A^{-1}(\theta_{t+1})\ [x_{t+1} - B(\theta_{t+1})\ (\hat{w}_t+w^*_t)],$$
$$\theta_t = b^*(\theta_{t+1},\hat{v}_t+v^*_t),$$
$$y_t = G(\theta_t)x_t + H(\theta_t)u_t.$$

where $(w^*_t, v^*_t)$ is the reverse-time MD-sequence of Theorem 5, $\hat{w}_t = f(t, \theta_{t+1}, \theta_t, x_{t+1})$, $\hat{v}_t = g(t, \theta_{t+1}, x_{t+1})$ and $f$, $g$ and $b^*$ are according to (11) and (13.b). The difference equation for $x_t$ is similar to the one for the linear Gaussian example in section 2.

But due to $\hat{w}_t$, it may even be nonlinear in $x_{t+1}$. At the end of the next section we will show that there are some further simplifications possible for this example, in case of in probability law equivalence.

## 5. EQUIVALENT IN LAW TIME-REVERSION

In this section we derive conditions under which the solutions of (1) and (4) are equivalent in law, and discuss these results for a jump-linear example. So far our line of reasoning is quite similar to the martingale approach of time-reversing a diffusion. However, things are quite different now we require equivalence in law only. The reason is that while in the diffusion situation this requires that $d\tilde{w}_t$ and $dw_t$ are equivalent in law, no similar simple results hold in the discrete-time situation. Instead of this, we identify the relation between conditional laws of $\tilde{w}_t$ and $w_t$ by a Bayesian estimation step. Next we characterize $f$ and the required law of $w^*_t$.

7 Theorem

Under assumption A.1 the solution $(\tilde{y}_t, \tilde{x}_t, \tilde{\theta}_t)$ of reverse-time system (4) is equivalent in law w.r.t. the solution $(y_t, x_t, \theta_t)$ of system (1) if,

(i) $P((\tilde{y}_T, \tilde{x}_T, \tilde{\theta}_T)\in dX) = P((y_T, x_T, \theta_T)\in dX)$; for any measurable $dX \subset R^m \times R^n \times M$,

(ii) $\tilde{a}$ and $\tilde{c}$ satisfy (13.a,c),

(iii) $P(\tilde{\theta}_t=\eta \mid \tilde{\theta}_{t+1}=\theta, \tilde{x}_{t+1}=x) = P(\theta_t=\eta \mid \theta_{t+1}=\theta, x_{t+1}=x)$,

(iv) $P(\tilde{w}_t\in dX \mid (\tilde{x}_{t+1}, \tilde{\theta}_{t+1}, \tilde{\theta}_t)=(x,\theta,\eta)] = P(w_t^*\in dX \mid (x_{t+1}, \theta_{t+1}, \theta_t)=(x,\theta,\eta))$,

all $(x,\theta,\eta,t)\in R^n \times M^2 \times [0,T-1]$ and measurable $dX \subset R^p$, with $w^*_t$ and $f$ satisfying (9.a), (10.a) and (11.a).

Proof: See Blom and Bar-Shalom (1989).

Our remaining problem is the characterization of the conditional law of $w^*_t$. As this is actually a discrete-time nonlinear filtering problem, it can be done by applying Bayes formula. We do this under the following additional assumptions:

A.1. The a priori distribution of $(x_t, \theta_t)$ permits a density-mass function for all $t\in[0,T]$.

A.4. $a^*(\theta,\eta,x,w)$ is once differentiable in $x\in R^n$ for all $(\theta,\eta,w)\in M^2 \times R^p$.

If the distributions in (iv) of Theorem 7 have density-mass functions then it can easily be verified that (iv) implies,

$$p_{w^*_t \mid \theta_{t+1}, \theta_t, x_{t+1}}(w \mid \cdot) = p_{w_t \mid \theta_{t+1}, \theta_t, x_{t+1}}(w+\hat{w}_t \mid \cdot), \quad (14.a)$$

where $\hat{w}_t$ satisfies (10.a).
With this our remaining step is to characterize the density at the right-hand side of (14.a) by applying Bayes formula.

8 Proposition
Under assumptions A.1 and A.4, the distribution in (iv) of Theorem 7 permits a density which is characterized by (14.a) and,

$$p_{w_t \mid \theta_{t+1}, \theta_t, x_{t+1}}(\cdot, \theta, \eta, x) = |\nabla_x a^{*T}(\theta, \eta, x, \cdot)| \cdot c(\theta,\eta,x)\, p_{w_t}(\cdot)\, p_{x_t \mid \theta_t}(a^*(\theta,\eta,x,\cdot) \mid \eta)], \quad (14.b)$$

with $\nabla_x$ the gradient and $c$ either a normalizing factor or zero iff $p_{x_{t+1} \mid \theta_{t+1}, \theta_t}(x \mid \theta, \eta) = 0$.
Moreover,

$$P(\tilde{\theta}_t=\eta \mid \tilde{\theta}_{t+1}=\theta, \tilde{x}_{t+1}=x) = P(\theta_t=\eta \mid \theta_{t+1}=\theta) \cdot p_{x_{t+1} \mid \theta_{t+1}, \theta_t}(x \mid \theta, \eta)\, p^{-1}_{x_{t+1} \mid \theta_{t+1}}(x \mid \theta). \quad (15)$$

Proof: See Blom and Bar-Shalom (1989).

Jump-linear example
For a linear system with first order Markovian switching coefficients we arrived, in section 4, at the following reversed-time equation:

$$x_t = A^{-1}(\theta_{t+1})[x_{t+1} - B(\theta_{t+1})[\hat{w}_t + w^*_t]],$$

with $w^*_t$ the reverse-time MD sequence and $\hat{w}_t = E(w_t \mid \theta_{t+1}, \theta_t, x_{t+1})$. Because $a^*$ is linear in $(x,w)$, its gradient w.r.t. $x$ is $w$-invariant, by which proposition 8 yields

$$p_{w_t \mid \theta_{t+1}, \theta_t, x_{t+1}}(w \mid \theta, \eta, x) = c_1(\theta,\eta,x)\, p_{w_t}(w)\, p_{x_t \mid \theta_t}(A^{-1}(\theta)[x - B(\theta)w] \mid \eta).$$

In spite of the simplification this is a form which is in general quite complex, by which $\hat{w}_t$ still may be a nonlinear function of $x_{t+1}$. Obviously, this type of complexity could have been expected, as it is well known that a discrete-time Bayesian estimation step leads to nonlinear equations, unless the prior densities involved are Gaussian.

## 6 CONCLUDING REMARKS

We considered the problem of reversing the Markov solution of a nonlinear stochastic difference equation in time. The nonlinearities were due to nonlinear coefficients and a hybrid state space, i.e. a product of an Euclidean space and a discrete set. For simplicity, it was assumed that the process in the discrete set satisfies the Markov property. Subsequently we gave a precise description of our time reversion objectives: the development of time reversed difference equations, of forms similar to the original equation, but driven by reversed-time martingale difference sequences, such that their solutions are respectively indistinguishable from and in

probability law equivalent to the solution of the original equation. Following this the derivation of the indistinguishable reverse-time equation was performed. The main new theoretical result is the introduction and evaluation of a non-canonical (Jacod and Shiryaev, 1987) reverse-time martingale decomposition, which is appropriate to the hybrid state space situation. In contrast with this, all previous reverse-time equations are based on a canonical martingale decomposition. After that, it was shown how the in probability law equivalent time reversed system can be obtained by introducing an appropriate Bayesian estimation step. As expected, this Bayesian estimation step leads to closed form equations whose dimensionality often complicates further applications. In view of this, in Blom and Bar-Shalom (1989) we elaborate the Bayesian step for linear systems with Markovian switching coefficients (jump-linear systems), and apply the the results to smoothing a trajectory with sudden manoeuvers.

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# A NEW CONTROLLER FOR DISCRETE-TIME STOCHASTIC SYSTEMS WITH MARKOVIAN JUMP PARAMETERS

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Abstract. A realistic stochastic control problem for hybrid systems with Markovian jump parameters may have the switching parameters in both the state and measurement equations. Furthermore, both the system state and the jump states may not be perfectly observed. Currently the only existing implementable controller for this problem is based upon a heuristic multiple model partitioning (MMP) and hypothesis pruning. In this paper we present a stochastic control algorithm for stochastic systems with Markovian jump parameters. The control algorithm is derived through the use of stochastic dynamic programming and is designed to be used for realistic stochastic control problems, i.e., with noisy state observations. The state estimation and model identification is done via the recently developed Interacting Multiple Model algorithm. Simulation results show that a substantial reduction in cost can be obtained by this new control algorithm over the (MMP) scheme.

Keywords. Stochastic control; Dynamic programming; Hybrid systems; Multiple model partitioning; Markovian jump parameters.

## 1. INTRODUCTION

An important problem of engineering concern is the control of discrete-time stochastic systems with parameters that may switch among a finite set of values. In this paper we present the development of a controller for discrete-time hybrid jump-linear Gaussian systems. Here the state and measurement equations have parameter matrices which are functions of a Markov switching process. The jump states are not observed and only the state is observed in the presence of noise.

Along with presenting a desirable practical control algorithm we also point out an interesting theoretical phenomenon. We show that there is a natural connection between the interacting multiple model (IMM) state estimation algorithm [B1] and the control of jump-linear systems. Thus the IMM is the state estimation algorithm of choice for use in these types of control problems.

Systems which pertain to the jump-linear modelling methodology are found in many areas. Systems of a highly nonlinear nature can be approximated by a set of linearized models [M3, V1, V2]. A failure in a component of a dynamical system (or subsequent repair) can be represented by a sudden change in the systems parameters [B2, S1, W1]. Also economic problems, which can be modelled by parameters that are subject to sudden changes due to shortages in important materials [G2]. And as is noted in [M6] there also exist applications to the design of control systems for large flexible structures in space.

There has been an extensive amount of work done in this area and on the related problem of controlling stochastic dynamic systems with unknown, time-invariant parameters. We refer the reader to the [T3] and [G3] for a list of references and a discussion of their scope and applications.

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More recently in [S2] a feedforward/feedback controller was presented for the continuous-time problem with a completely observed system state and where the "modal indicator" is measured with a high quality sensor. In [M6] the continuous-time jump-linear problem is considered where the system state and "modal processes" are perfectly observed. The optimal regulator was obtained and notions of stochastic stabilizability and detectability were introduced to characterize the behavior of the optimal system on long time intervals. In [M7] the continuous-time jump-linear problem with additive and multiplicative noises and noisy measurements of the plant state was considered with the plant mode assumed perfectly observed.

In [E1] a sufficient stability test is given for checking the asymptotic behavior of the error introduced by the averaging of hybrid systems. In [M8] the continuous-time jump-linear problem with non-Markovian regime changes was considered. A control scheme was presented for the case of perfect ovservations of the system state and plant regime.

In [C3] a discrete-time Markovian jump optimal control problem was considered. The controller is for the case of perfect system state observations and known form process. They derive necessary and sufficient conditions for the existence of optimal constant control laws which stabilize the controlled system as the time horizon becomes infinite. Through examples they show the interesting result that stabilizability of the system in each form is neither necessary nor sufficient for the existence of a stable steady-state closed-loop system.

In [Y1] a discrete-time system with perfect state and mode information was considered. A controller was presented which is stabilizing in the mean square exponential sense.

As pointed out in [G2], we generally cannot determine the optimal jump-linear quadratic Gaussian closed-loop control law analytically

even for a two-step problem. In order to compute the optimal control extensive numerical search methods must be employed and thus one would like to find simpler suboptimal control schemes.

Currently the only existing implementable controller for this problem (switching parameters in the system state and measurement equations and noisy state observations), is the one discussed in [T3] and is of the OLOF class. This algorithm is based upon a heuristic multiple model partitioning (MMP) and hypothesis pruning. The MMP approach, being simple and straightforward to implement, is a reasonable choice for the unknown parameter problem [L1], and as shown in [T3] it works well for applications involving switching parameters in the state measurement equation only. For the non-switching parameter problem the operating mode is determined to a high probability in a relatively short period of time and the MMP approach gives the linear quadratic Gaussian optimal control.

For switching parameter problems a different situation exists. Here because of the switching the operating mode may not be determined to a high probability. The proposed approach to deriving a suboptimal control scheme is to start with the solution to the optimal control problem via the use of stochastic dynamic programming. By utilizing dynamic programming and making appropriate suboptimal assumptions the use of numerical search methods has been avoided. We thus have developed a multiple model control scheme which has the following desirable properties: (a) it gives the optimal final control, (b) the algorithm utilizes the IMM state estimation scheme, and (c) it has the same property as the MMP approach in that it gives the optimal linear quadratic control under the assumption of a perfectly known model history sequence (which is however an unrealistic assumption for this class of problems).

For comparison purposes we implement the "switching parameters in the system state equation" controller, proposed (but not tested) in [T3]. We show via example that a statistically significant reduction in cost can be achieved through the use of our controller which also belongs to the OLOF class.

The paper is outlined as follows. In section 2 the problem formulation is given. In section 3 an interesting connection between the IMM state estimation algorithm and the control of multiple model systems is shown to exist. In section 4 we obtain the control algorithm. A new "full-tree" control algorithm is derived which utilizes all possible future parameter history sequences. In section 5 we use simulations to compare the MMP control algorithm with the full-tree controller.

## 2. PROBLEM FORMULATION

The problem to be solved, is discussed next. We took the pragmatic approach of starting with the available mathematical and statistical tools found to yield success in solving similar problems of this type in the past (i.e., use is made of the stochastic dynamic programming method and the total probability theorem, etc.). As we shall see, not only does this practical engineering approach yield an improved multiple model control algorithm, but it also leads to the interesting theoretical observation of a direct connection between the IMM state estimation algorithm and jump-linear control.

It is desired to find a sequence of causal control values to minimize the cost functional

$$J = E\{C(0)\} = E\left\{x(N)'Q(N)x(N) + \sum_{k=0}^{N-1}\left[x(k)'Q(k)x(k) + u(k)'R(k)u(k)\right]\right\} \quad (2.1)$$

where $Q(k) \geq 0$ for each $k=0,1,\ldots N$ and and it is sufficient that $R(k)>0$ for each $k=0,1,\ldots,N-1$.

The discrete-time system state and measurement modeling equations are

$$x(k) = F[M(k)]x(k-1) + G[M(k)]u(k-1) + v[k-1,M(k)] \quad (2.2a)$$

$$z(k) = H[M(k)]x(k) + w[k,M(k)] \quad k=0,1,2,\ldots \quad (2.2b)$$

where $x(k)$ is an $n\times 1$ system state vector, $u(k)$ is an $p\times 1$ control input, and $z(k)$ is an $m\times 1$ system state observation vector. The argument $M(k)$ denotes the model "at time $k$" - in effect during the sampling period ending at $k$. The process and measurement noise sequences, $v[k-1,M(k)]$ and $w[k,M(k)]$, are white and mutually uncorrelated.

The model at time $k$ is assumed to be among a finite set of $r$ models

$$M(k) \in (1,2,\ldots,r) \quad (2.3)$$

for example

$$F[M(k)=j] = F_j \quad (2.4)$$

$$v[k-1,M(k)=j] \sim N[\mu_j, V_j] \quad (2.5)$$

$$w[k,M(k)=j] \sim N[\lambda_j, \Psi_j] \quad (2.6)$$

i.e., the structure of the system and/or the statistics of the noises might be different from model to model.

The model switching process to be considered here is of the Markov type. The process is specified by a transition matrix with elements $p_{ij}$. Let

$$I^k \triangleq [z(0),z(1),\ldots,z(k),u(0),u(1),\ldots,u(k-1)] \quad (2.7)$$

denote the information available to the controller at time $k$ (i.e. the control is causal).

## 3. THE LAST STAGE CONTROL AND THE CONNECTION WITH THE IMM ESTIMATOR

An integral part of any control algorithm for this class of problems is the system state estimator. In this section we show that there exists an interesting connection between the control of multiple model stochastic systems and the IMM system state estimator [B1]. To this end we start by solving for the time N-1 optimal control. The optimal control at time N-1, is the value of $u(N-1)$ which minimizes

$$J(N-1) = E\left\{x(N-1)'Q(N-1)x(N-1) + u(N-1)'R(N-1)u(N-1) + x(N)'Q(N)x(N)\,\middle|\, I^{N-1}\right\}$$

$$= \sum_{j=1}^{r} E\left\{x(N-1)'Q(N-1)x(N-1) + u(N-1)'R(N-1)u(N-1) + x(N)'Q(N)x(N)\,\middle|\, I^{N-1}, M(N)=j\right\} \cdot P[M(N)=j|I^{N-1}] \quad (3.1)$$

$$\mu_j(N|N-1) \triangleq P[M(N)=j|I^{N-1}] \tag{3.2}$$

and use the state equation (2.2a) and (2.4), (2.5) in (3.1) to get

$$J(N-1) = \sum_{j=1}^{r} E\Big\{x(N-1)'\Big[Q(N-1)+F_j'Q(N)F_j\Big]x(N-1)$$

$$+2u(N-1)'G_j'Q(N)F_jx(N-1)+u(N-1)'\Big[R(N-1)+G_j'Q(N)G_j\Big]$$

$$\cdot u(N-1)\Big|I^{N-1},M(N)=j\Big\}\mu_j(N|N-1)$$

$$+\sum_{j=1}^{r} \mathrm{tr}[Q(N)V_1]\mu_j(N|N-1) \tag{3.3}$$

Now taking the partial of (3.3) w.r.t. u(N-1) and setting it to zero yields

$$u^*(N-1) = -\Big[R(N-1)+\sum_{j=1}^{r} G_j'Q(N)G_j\mu_j(N|N-1)\Big]^{-1}$$

$$\cdot \sum_{j=1}^{r} G_j'Q(N)F_jE\Big\{x(N-1)\Big|I^{N-1},M(N)=j\Big\}$$

$$\cdot \mu_j(N|N-1) \tag{3.4}$$

Notice that

$$E\Big\{x(N-1)\Big|I^{N-1},M(N)=j\Big\} = \sum_{i=1}^{r} E\Big\{x(N-1)\Big|I^{N-1},M(N)=j,$$

$$M(N-1)=i\Big\}P[M(N-1)=i|M(N)=j,I^{N-1}] \tag{3.5}$$

where, since M(N)=j in the first conditioning is irrelevant, the expectation inside the summation is

$$E\Big\{x(N-1)\Big|I^{N-1},M(N)=j\Big\} = \sum_{i=1}^{r} \hat{x}_i(N-1|N-1)\mu_{ij}(N-1|N-1)$$

$$\triangleq \hat{x}^{0j}(N-1|N-1) \tag{3.6}$$

which is the IMM mixed initial estimate [B1].

Thus using (3.6) in (3.4) we get

$$u^*(N-1) = -\Big[R(N-1)+\sum_{j=1}^{r} G_j'Q(N)\mu_j(N|N-1)\Big]^{-1}$$

$$\cdot \sum_{j=1}^{r} G_j'Q(N)F_j\hat{x}^{0j}(N-1|N-1)\mu_j(N|N-1) \tag{3.7}$$

## 4. THE CONTROL ALGORITHM

We will derive a full-tree control algorithm (FT) which computes control values by taking into account all possible future model histories. As will be seen by our example this method offers improved performance over the existing scheme [T3].

The l-th future *history of models* is denoted as

$$M^{k,l} = (M(k)=l_k,\ldots,M(N)=l_N) \qquad l=1,\ldots,r^{N-k+1} \tag{4.1}$$

where $l_i$ is the model at time i from history l and

$$1 \le l_i \le r \qquad i=k,\ldots,N \tag{4.2}$$

$$J^*(k,I^k) \triangleq \min_{u(k)} E\Big\{x(k)'Q(k)x(k)+u(k)'R(k)u(k)$$

$$+ J^*(k+1,I^{k+1})\Big|I^k\Big\} \tag{4.3}$$

where $J^*(k,I^k)$ is the optimal cost-to-go from time k to the end. Now applying the total probability theorem to (4.3) yields

$$J^*(k,I^k) = \min_{u(k)} \sum_{l=1}^{r^{N-k+2}} \Big[E\Big\{x(k)'Q(k)x(k)+u(k)'R(k)u(k)$$

$$+ J^*(k+1,I^{k+1})\Big|M^{k+1,l},I^k\Big\}P[M^{k+1,l}|I^k]\Big] \tag{4.4}$$

The control that minimizes an approximation to (4.4) is derived in the Appendix, and is given as

$$u^{FT}(k) = -\Big[R(k)+\sum_{l=1}^{r^{N-k+2}} G_{l_{k+1}}'P^l(k+1)G_{l_{k+1}}\mu_l(N|k+1)\Big]^{-1}$$

$$\cdot \sum_{l=1}^{r^{N-k+2}} G_{l_{k+1}}'P^l(k+1)F_{l_{k+1}}\hat{x}^{0l}(k|k)\mu_l(N|k+1) \tag{4.5}$$

and again we see the natural way the IMM mixed initial estimates show up.

Note that the control parameters $P^l(k)$ (model-history-conditioned optimal cost matrices) are computable off-line.

## 5. SIMULATION RESULTS

The FT controller developed in Sec. 4 is used to control the state trajectory of the system. The performance of this algorithm, as determined by (2.1), is compared to the cost obtainable by using the MMP controller discussed in [T3]. In order to obtain a meaningful comparison we use the rigorous statistical analysis technique presented in [B5, W3].

The control of a double integrator system with process and measurement noises is considered with a gain failure. The two possible models are given by the following system equation

$$x^i(k+1) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x^i(k) + \begin{bmatrix} 0 \\ b^i \end{bmatrix} u(k)$$

$$+ \begin{bmatrix} T^2/2 \\ T \end{bmatrix} v(k) \qquad i=1,2 \tag{5.1}$$

with measurement equation

$$z(k) = [1\ 0]\ x^i(k) + w(k) \tag{5.2}$$

The models differ in the control gain parameter $b^i$. The process and measurement noises are mutually uncorrelated with zero mean and variances given by

$$E[v(k)\ v(j)] = 0.16\ \delta_{kj} \tag{5.3}$$

and

$$E[w(k)\ w(j)] = \delta_{kj} \tag{5.4}$$

The control gain parameters were chosen to be $b^1=2$ and $b^2=0.5$.

The Markov transition matrix was selected to be

$$\begin{bmatrix} 0.1 & 0.9 \end{bmatrix} \tag{5.5}$$

For this example N=7, and the cost parameters R(k) and Q(k), (see (2.1)), were selected as

$$R(k) = 5.0 \qquad k=1,2,\ldots,N-1 \tag{5.6}$$

and

$$\begin{bmatrix} Q(0) \\ Q(1) \\ Q(2) \\ Q(3) \\ Q(4) \\ Q(5) \\ Q(6) \\ Q(7) \end{bmatrix} = \begin{bmatrix} \begin{matrix} 0.0 & 0.0 \\ 0.0 & 0.0 \end{matrix} \\ \begin{matrix} 2.0 & 0.0 \\ 0.0 & 2.0 \end{matrix} \\ \begin{matrix} 3.0 & 0.0 \\ 0.0 & 3.0 \end{matrix} \\ \begin{matrix} 4.0 & 0.0 \\ 0.0 & 4.0 \end{matrix} \\ \begin{matrix} 5.0 & 0.0 \\ 0.0 & 5.0 \end{matrix} \\ \begin{matrix} 6.0 & 0.0 \\ 0.0 & 6.0 \end{matrix} \\ \begin{matrix} 5.0 & 0.0 \\ 0.0 & 5.0 \end{matrix} \\ \begin{matrix} 20.0 & 0.0 \\ 0.0 & 8.0 \end{matrix} \end{bmatrix} \tag{5.7}$$

where the last matrix, Q(7), reflects our desire to drive $x_1(7)$ vigorously to zero. Also note that for this example T=1.0.

The real system was initialized with $x(0)=[30.0,\ 0.0]'$ and a random selection was done for choosing the initial model with $P[M(0)=i]=0.5$, $i=1,2$. The Kalman filters each received an initial state covariance of

$$P(0|0) = \begin{bmatrix} 1.0 & 1.0 \\ 1.0 & 2.0 \end{bmatrix} \tag{5.8}$$

and the initial state estimate was selected as

$$\begin{bmatrix} \hat{x}_1(0|0) \\ \hat{x}_2(0|0) \end{bmatrix} = \begin{bmatrix} z(0) \\ z(0) - z(-1) \end{bmatrix} \tag{5.9}$$

where $z(-1) = 30.0 + w(-1)$ and $z(0) = 30.0 + w(0)$.

Statistical tests were made on the results of 50 Monte Carlo runs. Sample means and variances of the Monte Carlo costs $C_i$ defined in (2.1) were computed for the FT, MMP, and "known model-history" (i.e. optimum linear quadratic) controllers.

Table I contains the results. The FT algorithm shows a clear reduction in cost as compared with the MMP scheme. However in order to provide a rigorous argument that the actual performance is ordered as Table I indicates we apply the statistical test presented in [BS, W3].

Table II contains the results. The sample standard deviation $\sigma_{\bar{\Delta}}$ of the mean of the cost differences, $C_i^{MMP}-C_i^{FT}$, are shown. The hypothesis that the FT controller is better than the MMP scheme can be accepted only if the probability of error $\alpha$ is less than, say, 1 percent. Then the threshold against which we compare the test statistic $\bar{\Delta}/\sigma_{\bar{\Delta}}$ is $\mu=2.33$. This test statistic has to exceed the threshold in order to accept the hypothesis.

TABLE I

SAMPLE AVERAGE COSTS AND STANDARD DEVIATIONS

| | Known Model-History | FT | MMP |
|---|---|---|---|
| Sample Mean | 2,647 | 6,063 | 19,519 |
| Sample Standard Deviation | 8,096 | 3.96E5 | 1.12E7 |

the FT controller performs better than the MMP controller for this problem. The estimated improvement (decrease in cost) of 70% is statistically significant.

TABLE II

STATISTICAL TEST FOR ALGORITHM COMPARISONS

| | $\bar{\Delta}$ | $\sigma_{\bar{\Delta}}$ | Test Statistic $\bar{\Delta}/\sigma_{\bar{\Delta}}$ | Estimated Improvement % |
|---|---|---|---|---|
| FT-MMP | 13,456 | 3,316 | 4.1 | 70 |

## 6. CONCLUSION

The development of a new control algorithm for discrete-time hybrid stochastic systems with Markovian jump parameters has been presented. This contoller was derived through the use of stochastic dynamic programming and by taking into account all possible future "histories of models". This scheme uses the IMM state estimation algorithm. We show that there is an interesting connection between the IMM state estimator and control of jump-linear hybrid systems. This new controller is of the OLOF class and has off-line computable control gain parameters.

From the example it is seen that this scheme can achieve a statistically significant reduction in cost when compared to the multiple model partitioning approach.

## APPENDIX

### I. Derivation of (4.5)

Note that given the future *history of* models $M^{k+1,N}$, the optimal cost-to-go $J^*(k+1,I^{k+1})$ is easily computed and is denoted.

$$J_i^*(k+1,I^{k+1}) \triangleq E\left\{x(k+1)'P^i(k+1)x(k+1)\middle|I^{k+1},M^{k+1,N}\right\} + \alpha^i(k+1) \tag{A.1}$$

where the notation from [B4] is used for $P(k+1)$ and $\alpha(k+1)$.

Since the expectation in (4.4) is conditioned on $M^{k+1,N}$, we obtain our of approximation by replacing $J^*(k+1,I^{k+1})$ inside the expectation with (A.1), and (4.4) becomes

$$J^*(k,I^k) \approx \min_{u(k)} \sum_{i=1}^{r^{N-k+2}} \left\{E\left\{x(k)'Q(k)x(k) + u(k)'R(k)u(k) + E\left\{x(k+1)'P^i(k+1)x(k+1)\middle|I^{k+1},M^{k+1,N}\right\} + \alpha^i(k+1)\middle|M^{k+1,N},I^k\right\}\mu_i(N|k+1)\right\} \tag{A.2}$$

where

$$\mu_i(N|k+1) \triangleq P(M^{k+1,N,j}|I^k| \qquad (A.3)$$

Now use (2.2a) and apply the smoothing property of expectation to (A.2) to get

$$J^*(k,I^k) \approx \min_{u(k)} \sum_{i=1}^{r^{N-k+2}} \Big\{ E\Big\{ x(k)'Q(k)x(k) + u(k)'R(k)u(k) + \Big[F_{i_{k+1}} x(k) + G_{i_{k+1}} u(k) + v[k-1,i_{k+1}|I\Big]' P^i(k+1)\Big[\cdot\Big] + \alpha^i(k+1)\Big|M^{k+1,N,j}, I^k\Big\}\mu_i(N|k+1)\Big| \qquad (A.4)$$

Take the partial w.r.t. u(k) of (A.4) and set to zero to solve for

$$u^*(k) = -\Big[R(k) + \sum_{i=1}^{r^{N-k+2}} G'_{i_{k+1}} P^i(k+1) G_{i_{k+1}} \mu_i(N|k+1)\Big]^{-1} \cdot \sum_{i=1}^{r^{N-k+2}} G'_{i_{k+1}} P^i(k+1) F_{i_{k+1}} E\Big\{x(k)\Big|M^{k+1,N,j}, I^k\Big\}\mu_i(N|k+1] \qquad (A.5)$$

We still need to evaluate the expectation in (A.5). This is done as follows. Note that x(k) is independent of M(i), i=k+2,....,N if M(k+1) is known, thus

$$E\Big\{x(k)\Big|M^{k+1,N,j}, I^k\Big\} = E\Big\{x(k)\Big|M(k+1) = i_{k+1}, I^k\Big\} \qquad (A.6)$$

But (A.6) is $\hat{x}^{0i}(k|k)$, the IMM mixed initial estimate (see (3.6)), thus using (A.6) in (A.5), we get (4.5).

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# FROM PIECEWISE DETERMINISTIC TO PIECEWISE DIFFUSION MARKOV PROCESSES

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ABSTRACT

Piecewise Deterministic (PD) Markov processes form a remarkable class of hybrid state processes because, in contrast to most other hybrid state processes, they include a jump reflecting boundary and exclude diffusion. As such, they cover a wide variety of impulsively or singularly controlled non-diffusion processes. Because PD processes are defined in a pathwise way, they provide a framework to study the control of non-diffusion processes along the same lines as that of diffusions. An important generalization is to include diffusion in PD processes, but, as pointed out by Davis, combining diffusion with a jump reflecting boundary seems not possible within the present definition of PD processes. This paper presents PD processes as pathwise unique solutions of an Itô stochastic differential equation (SDE), driven by a Poisson random measure. Since such an SDE permits the inclusion of diffusion, this approach leads to a large variety of piecewise diffusion Markov processes, represented by pathwise unique SDE solutions.

## 1. INTRODUCTION

Because many of the stochastic processes that we meet in nature have a state space that is a product of a continuous space and a discrete set, we often need pathwise models on such a hybrid state space. As a result, several classes of hybrid state space models have been developed, such as systems with Markovian switching coefficients, doubly stochastic counting processes and Markov decision drift processes. These models are used in quite different fields of applications, by which their studies have often evolved separately. One reason to study hybrid state space processes within a common framework is that their martingale parts are in general discontinuous. This property has attracted a lot of attention, and is by now very well documented (Jacod, 1979; Cinlar et al., 1980; Bremaud, 1981; Elliott, 1982; Bensoussan and Lions, 1984; Ethier and Kurtz, 1986; Jacod and Shiryaev, 1987). It is quite clear from these results that, to study hybrid state Markov processes along the same lines as diffusions, we need both pathwise representations and strong Markov (martingale) characterizations of those processes. Unfortunately, for hybrid state Markov processes there is presently a lacuna of pathwise representations with strong Markov characterizations. This lacuna is apparent if we depict the main classes of hybrid state Markov processes in the form of a Venn-diagram (fig. 1).

There exist pathwise representations with strong Markov characterizations of counting processes with diffusion intensity (Snyder, 1975; Marcus, 1978), of diffusions with Markovian switching coefficients (Wonham, 1970; Brockett and Blankenship, 1977) and of Piecewise Deterministic (PD) Markov processes (Davis, 1984). For many other Markov processes in figure 1, there exist only strong Markov characterizations (Kingman, 1975; Anulova, 1979, 1982; Bensoussan and Lions, 1984; Belbas and Lenhart, 1986). Actually, PD Markov processes seem the most interesting of all processes in figure 1, as they provide pathwise representations with a strong Markov characterization of all major non-diffusion Markov processes. As such, PD Markov processes provide a framework to study Markov decision drift processes (Hordijk and Van der Duyn Schouten, 1983; Yushkevich, 1983) along the same line as diffusions (Vermes, 1985). With this, an interesting generalization is to extend the spectrum of hybrid state Markov processes by including diffusion into PD Markov processes. As the present definition of PD processes does not seem to have an opening left for that inclusion (Davis, 1984), we need a different approach.

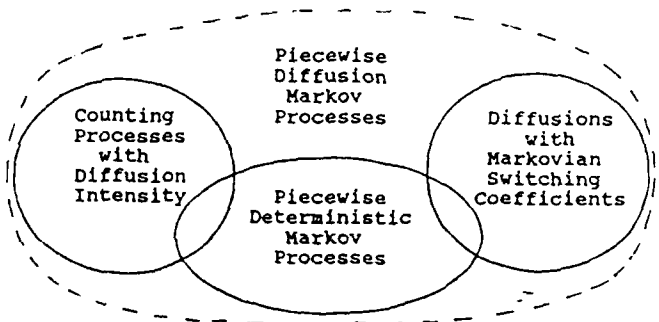


Fig. 1. Main classes of hybrid state Markov processes.

The approach that overcomes this difficulty, presented in the sequel, is to assume a stochastic differential equation (SDE) in a hybrid space and to construct a large class of piecewise diffusion Markov processes from it. With respect to the state space we restrict our attention to a hybrid subset of a Euclidean space. Then the most general SDE is of Itô type, driven by Brownian motion, $w$, and a Poisson random measure, $p$ on $(0,\infty)\times U$,

$$d\xi_t = \alpha(\xi_t)dt + \beta(\xi_t)dw_t + \int_U \psi(\xi_{t-},u)\, p(dt,du).$$

The path of a solution of this SDE is right continuous and has left hand limits: $\xi_{t-} = \lim_{\Delta \downarrow 0} \xi_{t-\Delta}$. If $p$ generates a multivariate point $(t,u_t)$, then the path of $\xi$ has a discontinuity:

$$\xi_t = \xi_{t-} + \psi(\xi_{t-},u_t).$$

In the sequel we shall focus on pathwise unique solutions. The classical result for the existence of such solutions requires that $\psi$ is sufficiently continuous (Gihman and Skorohod, 1972), which restricts the SDE essentially to systems with Markovian switching coefficients. However, there are some non-classical pathwise uniqueness results that allow a discontinuous $\psi$ (Lepeltier and Marchal, 1976; Jacod and Protter, 1982; Veretennikov, 1988). Taking these results as a starting point, we introduce and evaluate a particular structure for $\psi$ in section 2. This structure poses hardly any restrictions on the possible solution of the SDE, while it enables a separate evaluation of an unbounded jump intensity and a hybrid state space situation. In view of this separation, we first consider, in sections 3 and 4, the modelling of a jump reflecting boundary in $\mathbb{R}^n$ through an unbounded jump intensity, and after that, in section 5, we consider the hybrid state situation.
Assume an open subset $O$ of $\mathbb{R}^n$ with jump reflecting boundary $\partial O$, which means that $(\xi_t)$ undergoes an

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instantaneous jump into the interior of $0$ if $(\xi_t)$ tries to cross or to travel through $\partial 0$. To model this with the above SDE, the Poisson random measure $p$ should instantaneously generate a point when $(\xi_t)$ enters $\partial 0$. However, this is not possible as a Poisson random measure generates almost surely no point at an entrance time. To overcome this problem, we briefly discuss the following three approaches:

1. Replace $p$ by a random measure, with almost surely one point at an arbitrary time.
2. Assume a $\psi$ such that $p$ generates an active point during an infinitesimal small time interval after entering $\partial 0$.
3. Assume a $\psi$ such that $p$ generates an active point during an infinitesimal small time interval just before entering $\partial 0$.

Approach 1 adequately solves the instantaneous jump problem but creates many new problems, because if $p$ is not a Poisson random measure, then the SDE can not be analysed within the powerful Itô framework. Approach 2 is the well known approach of randomized stopping (Bensoussan and Lions, 1984). As this approach allows $(\xi_t)$ to cross or to travel through $\partial 0$ the resulting process is at best a modification of a PD Markov process. Approach 3 is the desired solution. However, the problem with approach 3 is that it is in general not known how to carry it out. A constructive answer to this will be given in the sequel. It is clear that approach 3 needs a kind of prediction of the time that $(\xi_t)$ might, otherwise, enter $\partial 0$. Actually, PD Markov processes are presently the only processes for which this prediction problem is solved (Davis, 1984). As such, we first formulate that solution in an SDE set up in section 3. Next, in section 4, we present a solution of the prediction problem for the situation with diffusion.

Finally, in section 5, we explicitly consider the hybrid state space situation. The most interesting effect of the hybrid state space assumption is that it leads to a particular type of jumps: jumps in the continuous state component of $(\xi_t)$ that anticipate a simultaneous transition of the discrete component of $(\xi_t)$. This type of jumps have been introduced by Gnedenko and Kovalenko (1968) for piecewise linear processes and by Sworder (1972) for systems with Markovian switching coefficients. For short we refer to these anticipating simultaneous jumps as hybrid jumps. The SDE framework of this paper provides an elegant way of representing the hybrid jumps of PD Markov processes and their piecewise diffusion generalizations.

Some other interesting generalizations of PD Markov processes, not considered in the sequel, are the inclusion of continously reflecting or sticky boundaries. The inclusion of a continuously reflecting boundary, while preserving pathwise uniqueness, seems possible if that boundary is smooth enough (Chaleyat-Maurel et al., 1980; Menaldi and Robin, 1985; Frankowska, 1985; Saisho, 1987). The inclusion of a sticky boundary without loosing pathwise uniqueness seems difficult if not impossible, but strong Markov characterizations are possible (Kingman, 1975; Anulova, 1979, 1982).

Notation

$R^+ = (0,\infty)$ and $R^- = (-\infty,0)$,
$R_+ = R^+ + (0)$ and $R_- = R^- + (0)$.
$Z = (..,-2,-1,0,1,2,..)$,
$N = (1,2,3,..)$.
$\underline{\xi} = \mathrm{Col}(\xi_2,..,\xi_n)$ if $\xi = \mathrm{Col}(\xi_1,..,\xi_n)$

$\|\sigma\|^2 = \sum_{i,j} \sigma_{ij}^2$, if $\sigma$ is a matrix
$|\sigma|^2 = \sum_i \sigma_i^2$, if $\sigma$ is a vector

$\xi^i_t$ : i-th component of process $\xi_t$.
$\partial 0$ : boundary of the closure of set $0$.
Int(x) : integer part of x.
$\chi$ : $\chi$(True)=1 and $\chi$(False)=0.
CADLAG : right continuous with left hand limits
$C^k(0)$ : the set of all real-valued functions that are k times continuously differentiable on $0$. The superscript is deleted if k=0. If k is followed by b, then f and its first k derivatives are bounded on $0$.
$\mathcal{D}(\mathcal{A})$ : domain of operator $\mathcal{A}$.

## 2 THE SDE OF LEPELTIER AND MARCHAL

We assume a stochastic basis $(\Omega,\mathcal{F},\mathbf{F},P)$, endowed with an m-dimensional standard Wiener process, $(w_t)$, and a Poisson random measure, $p(dt,du)$ on $R_+ x R^{d+1}$ (Jacod and Shiryaev, 1987, p. 70), with intensity measure $dt \times m(du)$, and consider the following stochastic differential equation (SDE) in $R_+ x R^n$,

$$d\xi_t = \alpha(\xi_t)dt + \beta(\xi_t)dw_t + \int_{R^- x R^d} \psi(\xi_{t-},u)\, q(dt,du) + \int_{R^+ x R^d} \psi(\xi_{t-},u)\, p(dt,du) , \qquad (1)$$

where $q$ is the martingale measure of $p$, $\xi_0$ is an $\mathcal{F}_0$-measurable random variable, while $\alpha$, $\beta$ and $\psi$ are measurable mappings of appropriate dimensions.

The classical reference for equation (1) is Gihman and Skorohod (1972). Significant extensions of their results have been obtained by Lepeltier and Marchal (1976) in their study of the relation between an integro-differential operator and an SDE of type (1). Their particular SDE can easily be obtained from (1), by introducing homeomorphism mappings of $R^- x R^d$ into $(u \in R^{d+1}; 0<|u|\le 1)$ and of $R^+ x R^d$ into $(u \in R^{d+1}; 1\le|u|<\infty)$, and subsequently transforming $m$ and $\psi$ correspondingly. Consequently, the results of Lepeltier and Marchal can immediately be used in the present study of (1), while allowing the intensity of the active points in $R^+$ to be unbounded outside some known Borel set $0' \subset R^n$.

Assumptions

A.1 There is a constant K such that, for all $\xi \in R^n$,
$|\alpha(\xi)|^2 + \|\beta(\xi)\|^2 + \int_{R^- x R^d} |\psi(\xi,u)|^2 m(du) \le K(1+|\xi|^2)$.

A.2 For all $k \in N$ there exists a constant $L_k$ such that, for all $\xi$ and $y$ in the ball $B_k=(u \in R^n; \|u\|^2 \le k)$,
$|\alpha(\xi)-\alpha(y)|^2 + \|\beta(\xi)-\beta(y)\|^2 + \int_{R^- x R^d} |\psi(\xi,u)-\psi(y,u)|^2 m(du) \le L_k|\xi-y|^2$.

A'.3 $0'$ is a known Borel subset of $R^n$, $\int_{R^+ x R^d} \chi(\psi(\xi,u)\neq 0)\, m(du)$ is uniformly bounded on $0'$, and $[\xi+\psi(\xi,u)] \in 0'$, for all $\xi \in R^n$, $u \in R^{d+1}$.

A'.4 For all $k \in N$ there exists a constant $M_k$, such that, with $B_k$ the ball of A.2:

a. for all $\xi \in B_k \cap 0'$,
$\int_{R^+ x R^d} |\psi(\xi,u)|\, m(du) \le M_k$.

b. for all $\xi \in B_k \cap (R^n - 0')$,
$\int_0^1 \int_{R^d} |\psi(\xi,u)|\, m(du) \le M_k$,
given that, for all $u \in R^+ x R^d$, $\psi(\xi,u) = \psi(\xi,u+\mathrm{Col}(1,0,..,0))$.

A'.5 For all $\tau \in N$ there is a constant $N_\tau$, such that
$E\left( \int_0^\tau \int_{R^+ x R^d} \chi(\psi(\xi_{s-},u)\neq 0)\, p(ds,du) \right) \le N_\tau$.

2.1 Proposition
Given $m(du)=du_1 x \mu(du)$ and assumptions A.1, A.2, A'.3, A'.4, A'.5 are satisfied. Then equation (1) has for any $\xi_0 \in 0'$ a pathwise unique solution, $(\xi_t)$. Moreover $(\xi_t)$ is then a right continuous Markov process.

Remark: Proposition 2.1 is a version of Theorem $III_4$ of Lepeltier and Marchal (1976), in the sense that they considered the situation of $0' = R^n$. Nevertheless, for the proof we can almost follow Lepeltier and Marchal. Another recent extension of Theorem $III_4$ of Lepeltier and Marchal is to the situation of a non-Lipschitzian $\alpha$ in turn of a sufficient non-degeneracy assumption on $\beta$ (Veretennikov, 1988).

Proof:
If (1)'s fourth right hand term vanishes, then it is well known that A.1 and A.2 are sufficient conditions (Gihman and Skorohod, 1972). As such, we have to show that (1)'s fourth right hand term does not change that situation, under A'.3, A'.4 and A'.5.
Due to A'.3 and the definition of Itô integration a solution of (1) is CADLAG. Due to A'.5, the discontinuities in $(\xi_t)$, that are caused by (1)'s fourth right hand term, are countable. Therefore we can associate with each discontinuity a time, $T_i$,

and a multi-variate point, $u_{T_i}$, such that $0<T_1<T_2<..<T_i<..$ and $\lim_{i\to\infty} T_i = \infty$. Due to the latter and $(\xi_t)$ being CADLAG,

$$\int_0^t \int_{R^+\times R^d} \psi(\xi_{s-},u)p(dt,du) = \sum_{0<T_i\le t} \psi(\xi_{T_i-},u_{T_i}).$$

If (1)'s first three right hand terms vanish, then the latter sum is finite (a.s.) for all $t\in R_+$, due to A'.4 and A'.5. With this result it is sufficient to show that (1) has a pathwise unique solution on an arbitrary finite time-interval $[0,T]$. For the existence of a solution, see the proof of Th. III$_4$ of Lepeltier and Marchal (1976; pp. 82-85). We already know that a solution is unique and $\mathcal{F}_t$-measurable on $[0,T_1)$. Because $\xi_t$ is CADLAG and $\psi$ is measurable, $T_1$ is $\mathcal{F}_{T_1}$-measurable. Then, by the definition of a Poisson random measure (Jacod and Shiryaev, 1987, pp. 65-66) $u_{T_1}$ is $\mathcal{F}_{T_1}$-measurable $\Rightarrow$ $\xi_{T_1} = \xi_{T_1-} + \psi(\xi_{T_1-},u_{T_1})$ is $\mathcal{F}_{T_1}$-measurable and, due to A'.3, $\xi_{T_1}\in 0$ $\Rightarrow$ Pathwise uniqueness holds true on $[0,T_1]$ and $\xi_{T_1}\in 0$. Due to the latter, we can repeat the procedure to show that pathwise uniqueness holds true on $[T_1,T_2]$ and $\xi_{T_2}\in 0$, and so on for the countable sequence of intervals. Q.E.D.

The interesting aspect of proposition 2.1 is, that the coefficients of (1)'s fourth right hand term may be discontinuous in $\xi$. This is exactly what we need, to construct a class of hybrid state Markov processes that is larger then the class of solutions of systems with Markovian switching coefficients. The first step towards this construction is replacing $\psi(\xi,u)$ by

$$\psi'(\xi,u) = \psi(\xi,u)\ X(\ [u_1<\lambda(\xi)]\ \cup\ [F(\xi)\neq 0]\ ), \quad (2.a)$$

where F is a measurable mapping of $R^n$ into $(0,1)$, $\psi$ and $\lambda$ are measurable mappings of appropriate dimensions, while the range of $\lambda$ is $R_+$. With this (1) becomes

$$d\xi_t = \alpha(\xi_t)dt + \beta(\xi_t)dw_t + \int_{R^-\times R^d} \psi(\xi_{t-},u)\ q(dt,du) + \int_{R^+\times R^d} \psi'(\xi_{t-},u)\ p(dt,du). \quad (2.b)$$

Assumptions

A.1 Define $0' \equiv \{\xi\in R^n;\ F(\xi)=0\}$, $\{\xi+\psi(\xi,u)\}\in 0'$, for all $\xi\in R^n$, $u\in R^{d+1}$.

A".4 Given, for all $\xi\in R^n-0'$ and $u\in R^+\times R^d$,
$\lambda(\xi)=1$,
$\psi(\xi,u)=\psi(\xi,u+Col(1,0,..,0))$,
and for any $k\in N$ there exists a constant $M_k$, such that
$\int_0^{\lambda(\xi)} \int_{R^d} |\psi(\xi,u)|\ m(du) \le M_k$, for all $\xi\in B_k$.

A".5
a. $\lambda(\xi)$ is on $0'$ uniformly bounded and continuous in $\xi$.
b. $(\xi_t)$, $t\in R_+$, exits $0'$ at most a countable number of times.

2.2 Theorem
Given $m(du)=du_1\times\mu(du)$ and assumptions A.1, A.2, A.3, A".4, A".5 are satisfied. Then equation (2.a,b) has for any $\xi_0\in 0'$ a pathwise unique solution $(\xi_t)$. Moreover $(\xi_t)$ is then a Markov process, of which the sample paths are measurable on the stochastic basis $(\Omega,\mathcal{F},F,P)$.

Proof:
Because, on $0'$, $\lambda(\xi)$ is continuous in $\xi$ (due to A".5,a) and $X(u_1<\lambda')$, $\lambda'\in R$, defines a measurable mapping of $R^2$ into $(0,1)$ $\Rightarrow$ $X(u_1<\lambda(\xi))$ defines a measurable mapping of $R\times 0'$ into $(0,1)$. Because the range of F is $(0,1)$, we can write
$X(\ [u_1<\lambda(\xi)]\ \cup\ [F(\xi)\neq 0]\ ) = X(\ u_1<\lambda(\xi)\ ) \vee F(\xi)$, of which both right hand terms are measurable. This implies that the supremum is measurable, which combined with the measurability of $\psi$, makes that $\psi'$ is measurable. This ensures that (2.b) is a special case of (1), with $\psi$ replaced by $\psi'$ according to (2.a). With this we are left to verify that A.3, A".4 and A".5 guarantee that A'.3, A'.4 and A'.5 are satisfied, which is straightforward. Q.E.D.

Having theorem 2.2, we are prepared to consider a jump reflecting boundary (in sections 3 and 4) and the hybrid state space situation (in section 5). But first we give a strong Markov characterization of $(\xi_t)$ if there is no reflecting boundary.

2.3 Proposition
Given F vanishes everywhere and the assumptions of theorem 2.2 are satisfied. Then for all $\xi_0\in R^n$, $(\xi_t)$ is a semi-martingale strong Markov process, and its extended generator, $\mathcal{A}$, is given by:

$$\mathcal{A}f = \mathcal{L}f + \mathcal{J}^-f + \mathcal{J}^+f, \quad \text{for all } f\in C^{2,b}(R^n), \quad (3)$$

where

$$\mathcal{L}f(\xi) = \sum_{i=1}^n \alpha_i(\xi) f_{\xi_i}(\xi) + \tfrac{1}{2}\sum_{i,j=1}^n [\beta(\xi)\beta(\xi)^T]_{ij} f_{\xi_i\xi_j}(\xi), \quad (4)$$

$$\mathcal{J}^-f(\xi) = \int_{R^n-\{0\}} [f(\xi+\zeta)-f(\xi)-\sum_{i=1}^n \zeta_i f_{\xi_i}(\xi)]\ S^-(\xi,d\zeta), \quad (5)$$

$$\mathcal{J}^+f(\xi) = \int_{R^n-\{0\}} [f(\xi+\zeta)-f(\xi)]\ S^+(\xi,d\zeta), \quad (6)$$

and for all Borel $A\subset R^n-\{0\}$,

$$S^-(\xi,A) \equiv \int_{R^-\times R^d} X[\ \psi(\xi,u)\in A\ ]\ m(du), \quad (7)$$

$$S^+(\xi,A) \equiv \int_{0+}^{\lambda(\xi)} \int_{R^d} X[\ \psi(\xi,u)\in A\ ]\ du_1\ \mu(du). \quad (8)$$

Proof:
Due to A.3, A".4, A".5 and $0'=R^n$, the $\mathcal{F}_t$-predictable part of $\xi_t$ is

$$A_t = \int_0^t \alpha(\xi_s)ds + \int_0^t \int_0^{\lambda(\xi_{s-})} \int_{R^d} \psi(\xi_{s-},u)\ m(du)ds.$$

Obviously, $(A_t)$ is of finite variation on any finite time-interval, while $(\xi_t-A_t)$ is a local martingale $\Rightarrow$ $(\xi_t)$ is a (special) semimartingale (Jacod and Shiryaev, p.43, Def. 4.21). This immediately implies that $(\xi_t)$ is a strong Markov process. Because $(\xi_t)$ is a semimartingale, the generator $\mathcal{A}$ follows from Itô's differentiation rule for discontinuous semimartingales (Elliott, 1982). Q.E.D.

## 3. PIECEWISE DETERMINISTIC MARKOV PROCESSES

In this section, we represent PD Markov processes as solutions of an SDE. Therefore, we consider (2.a,b) with $\beta=0$ and $\psi$ vanishing on $R^-\times R^d$:

$$d\xi_t = \alpha(\xi_t)dt + \int_{R^+\times R^d} \psi(\xi_{t-},u)\cdot X(\ [u_1<\lambda(\xi)]\ \cup\ [F(\xi)\neq 0]\ )\ p(dt,du), \quad (9)$$

Our goal is to introduce a particular mapping $F:R^n\to(0,1)$, such that (9) has pathwise unique solutions which are PD Markov processes. The present definition of a PD Markov process (Davis, 1984) works without such a mapping F. Instead, there is given an open subset 0 of $R^n$, with a jump reflecting boundary $\partial 0$, such that $(\xi_t)$ instantaneously jumps into the interior of 0 just before it would, otherwise, cross or travel through $\partial 0$. For the definition of a PD Markov process from (9) an appropriate F has to be constructed from 0 and $\alpha$. The construction of F will be based on the following differential equation, on $(0,\infty)\times R^n$,

$$d\xi'_t = \alpha(\xi'_t)dt, \qquad t\in(0,\infty), \quad (10)$$

which has pathwise unique solutions, assuming that $\alpha$ satisfies conditions A.1 and A.2. From this, we define $\partial\underline{0}$ as the set containing all elements of $\partial 0$ that are directly accessible by $(\xi'_t)$ from 0:

$$\partial\underline{0} \equiv \{\xi\in\partial 0\ ;\ \exists\ \tau\in(0,\infty) \text{ and } \xi'_0\in 0 \text{ such that } \xi'_\tau=\xi\ \wedge\ \xi'_{\tau-}\in 0\}. \quad (11)$$

Next we introduce the following distance function,

$$d_\alpha(\xi,\partial\underline{0}) \equiv \inf\{\tau\ge 0\ ;\ \xi'_0=\xi\ \wedge\ \xi'_\tau\in\partial\underline{0}\}, \quad (12)$$

which is, under the above mentioned conditions on $\alpha$, a measurable mapping of $R^n$ into R. With this we define, for $i\in N$,

$$0_i = \{\xi\in 0\ ;\ d_\alpha(\xi,\partial\underline{0}) \ge 1/i\}, \quad (13)$$

which are then Borel sets, and which form the Borel set

$$0' \equiv \bigcup_{i\in N} 0_i. \quad (14)$$

Now we define our particular F as follows:

$$F(\xi) = 1 \text{ , if } \xi\in R^n-0', \quad = 0 \text{ , else.} \quad (15)$$

Due to the above construction, F is measurable, by which theorem 2.2 yields:

3.1 Corollary
Given an open subset 0 of $R^n$, and a mapping F, defined by (10) through (15). Then, under the assumptions of theorem 2.2, equation (9) has for any

$\xi_0 \in O'$ a pathwise unique solution $(\xi_t)$. Moreover, $(\xi_t)$ is then a Markov process, of which the sample paths are measurable on the stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, P)$.

Next, we come to the main result of this section, which implies that $(\xi_t)$ is a Piecewise Deterministic Markov process.

3.2 Theorem
With probability one, the process $(\xi_t)$, of corollary 3.1, exits $O \cup \partial O$ zero times on $(0,\infty)$.

Proof:
By the definition of F, all points of p in $R^+$ become active as soon as $(\xi_t)$ has exit $O'$. This situation holds on until $(\xi_t)$ reenters $O'$. The reentering may occur due to drift or due to a jump generated by a point of p in $R^+$. Obviously, the cases that $(\xi_t)$ reenters $O'$ by drift without exit of $O \cup \partial O$ do not cause any difficulties. In all other cases, the probability of exit $O \cup \partial O$ by drift is

$$\int_{\kappa}^{\infty} \exp(-s/\tau)\, ds = \tau \exp(-\kappa/\tau),$$

with $\kappa = \inf(1/i \,;\, i \in N)$ and $1/\tau$ the intensity of points of p in $R^+$. Because $(\xi_t)$ exits $O'$ at most a countable number of times, the probability of exit $O \cup \partial O$ at least once is then $\tau/\kappa \exp(-\kappa/\tau)$. If all points of p in $R^+$ are active, then because $\kappa \in N$,

$$\lim_{\tau \downarrow 0} \tau/\kappa \exp(-\kappa/\tau) = 0,$$

which means a zero probability to exit $O \cup \partial O$ on $(0,\infty)$. Q.E.D.

3.3 Theorem
The process $(\xi_t)$, of corollary 3.1, is a semimartingale strong Markov process, and its extended generator, $\mathcal{A}$, is given by:

$$\mathcal{A}f = \mathcal{L}f + \mathcal{J}^+ f, \quad \text{for all } f \in \mathcal{D}(\mathcal{A}),$$

where $\mathcal{L}$ and $\mathcal{J}^+$ are given in proposition 2.3 with $\beta = 0$, while the domain of $\mathcal{A}$ is:

$$\mathcal{D}(\mathcal{A}) = \{ f \in C^{1,b}(O) \cap C^b(O \cup \partial O) \,;\, \mathcal{J}^+ f(\xi) = 0, \text{ all } \xi \in \partial O \}.$$

Proof:
Define a process $A_t$ as follows:

$$A_t = \int_0^t \alpha(\xi_s) ds + \int_0^t \chi(\xi_{s-} \in O') \int_0^{\Lambda(\xi_{s-})} \int_R d\, \psi(\xi_{s-}, u) \cdot m(du) ds + \sum_{i \geq 1} \int_0^t \int_R d\, \psi(\xi_{S_i -}, u)\, du_1 x_\mu(d\underline{u}),$$

with $S_i$ the $\mathcal{F}_t$-adapted times that $(\xi_t)$ jumps from $R^n - O'$ into $O'$, $i \geq 1$ and $S_0 \leq 0$,

$$S_i = \inf_{s > 0} \{ s > S_{i-1} \,;\, \xi_{s-} \in R^n - O' \wedge \xi_s \in O' \}.$$

Obviously, $(A_t)$ is of finite variation on any finite time-interval, while $(\xi_t - A_t)$ is a local $\mathcal{F}_t$-martingale. Subsequently, $(\xi_t)$ is a semimartingale. Application of Itô's differentiation rule for discontinuous (piecewise deterministic) semimartingales to $f(\xi_t)$, with $f \in C^1$, yields:

$$f(\xi_t) = f(\xi_0) + \sum_{i=1}^{n} \int_0^t \frac{\partial}{\partial \xi_i} f(\xi_{s-})\, [d\xi_s]_i + \sum_{0 < s \leq t} \int_R d\, [f(\xi_{s-} + \psi(\xi_{s-}, u)) - f(\xi_{s-})] - \sum_{i=1}^{n} \frac{\partial}{\partial \xi_i} f(\xi_{s-})\, [\psi(\xi_{s-}, u]_i)\, p((s), du),$$

up to indistinguishability.
Substitution of A".4,

$$p(ds, du) = q(ds, du) + ds \times m(du),$$
$$d\xi_s = dA_s + d(\text{local martingale}),$$
$$m(du) = du_1 x_\mu(d\underline{u}),$$

and using $f \in C^{1,b}(0) \cap C^b(O \cup \partial O)$, yields

$$f(\xi_t) = f(\xi_0) + \sum_{i=1}^{n} \int_0^t \frac{\partial}{\partial \xi_i} f(\xi_s)[\alpha(\xi_s)]_i ds + \int_0^t \chi(\xi_{s-} \in O') \int_0^{\Lambda(\xi_{s-})} \int_R d\, [f(\xi_{s-} + \psi(\xi_{s-}, u)) - f(\xi_{s-})]\, ds x du_1 x_\mu(d\underline{u}) + \sum_{i \geq 1} \int_0^t \int_R d\, [f(\xi_{S_i -} + \psi(\xi_{S_i -}, u)) - f(\xi_{S_i -})]\, du_1 x_\mu(d\underline{u}) + d(\text{local martingale}),$$

up to indistinguishability.
Next we use the property that

$$\mathcal{J}^+ f(\xi) = 0, \quad \text{all } \xi \in \partial O.$$

Because $\alpha$ is of linear growth and $(\xi_t)$ is locally bounded, $(\alpha(\xi_t))$ is locally bounded. This implies that $(\xi_t)$ does not increase while travelling through $O - O'$ to $\partial O$, as this takes a time interval of zero duration. The latter and the assumptions that $f \in C^b(O \cup \partial O)$ and $\mathcal{J}^+ f(\xi) = 0$ for all $\xi \in \partial O$, imply that $\mathcal{J}^+ f(\xi_s) = 0$ for all $\xi_s \in O - O'$. With this,

$$f(\xi_t) = f(\xi_0) + \int_0^t \mathcal{L} f(\xi_s)\, ds + d(\text{local martingale}) + \sum_{0 < s \leq t} \int_0^{\Lambda(\xi_{s-})} \int_R d\, [f(\xi_{s-} + \psi(\xi_{s-}, u)) - f(\xi_{s-})]\, ds x du_1 x_\mu(d\underline{u}).$$

Substitution of $\mathcal{J}^+$ yields

$$f(\xi_t) = f(\xi_0) + \int_0^t \mathcal{A} f(\xi_s)\, ds + d(\text{local martingale}),$$

which implies that $(\xi_t)$ is a strong Markov process with extended generator $[\mathcal{A}, \mathcal{D}(\mathcal{A})]$. Q.E.D.

## 4. PIECEWISE DIFFUSION MARKOV PROCESSES

Having obtained PD Markov processes as solutions of an SDE, the next step is to include diffusion. Therefore we consider the following SDE:

$$d\xi_t = \alpha(\xi_t) dt + \beta(\xi_t) dw_t + \int_{R^+ \times R^d} \psi(\xi_{t-}, u) \cdot \chi(\{u_1 < \Lambda(\xi)\} \cup \{F(\xi) \neq 0\})\, p(dt, du), \quad (16)$$

which corresponds to (II.a,b) if $\psi$ vanishes on $R^- \times R^d$. Initially we assume that $\beta(\xi)\beta(\xi)^T$ is positive definite for all $\xi \in R^n$, but relax this assumption further on.
Now we construct F, starting from the following differential equation, on $(0,\infty) \times R^n$,

$$d\xi'_t = \alpha(\xi'_t) dt + \beta(\xi'_t) dw_t, \qquad t \in (0,\infty), \quad (17)$$

which has pathwise unique solutions under assumptions A.1 and A.2, and which defines a family of homogeneous Markov processes with a measurable transition function

$$P'_t(\tau, A) \equiv P(\xi'_{\tau} \in A \,|\, \xi'_0 = \xi), \quad \text{all Borel A.} \quad (18)$$

Because $\beta\beta^T$ is positive definite, any element of $\partial O$ is accessible by $(\xi'_t)$ from O. Therefore we initially use the following Euclidean distance function,

$$d_\beta(\xi, \partial O) \equiv \inf \{ |\xi - y| \,;\, y \in \partial O \}, \quad (19')$$

which, obviously, is a measurable mapping.
Next, we define the Borel sets $O_i$ as follows,

$$O_i = \{ \xi \in O \,;\, d_\beta(\xi, \partial O) \geq 1/i \}, \qquad i \in N, \quad (20)$$

and from this the Borel set

$$O' \equiv \bigcup_{i \in N} O_i. \quad (21)$$

As before, we define our particular F as follows:

$$F(\xi) = 1 \text{ , if } \xi \in R^n - O', \quad = 0 \text{ , else.} \quad (22)$$

Obviously, F is measurable, by which theorem 2.2 yields:

4.1 Corollary
Given an open subset O of $R^n$, and a mapping F, defined by (17), (18), (19'), (20), (21) and (22). Then, under the assumptions of theorem 2.2, equation (16) has for any $\xi_0 \in O'$ a pathwise unique solution $(\xi_t)$. Moreover, $(\xi_t)$ is then a Markov process, with sample paths being measurable on the stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, P)$.

Next, we come to the characterization of the boundary behaviour and the strong Markov property of $(\xi_t)$.

4.2 Theorem
With probability one, process $(\xi_t)$, of corollary 4.1, exits $O \cup \partial O$ zero times on $(0,\infty)$.

Proof:
By the definition of F, all points of p become active as soon as $(\xi_t)$ has exit $O'$, say at moment T, which situation continues until $(\xi_t)$ has reentered $O'$, say at moment $T + \Delta$. The exit may occur due to diffusion or due to a jump generated by a point of p in $R^+$. Obviously, the cases that $(\xi_t)$ exits $O - O'$ by diffusion without entering $\partial O$ do not cause any difficulties. In all other cases we know from the proof of theorem 3.3 that $\Delta$ has an exponential distribution of which both the mean and the standard deviation equals $\tau \downarrow 0+$. With this, it follows that, for any $\xi \in O'$, the probability of entering and exiting $\partial O$ within $1/\tau$ is:

$$\tau^{-1} P'_t(\tau, R^n - O - \partial O) \leq \tau^{-1} P_t'(\tau, (y \in R^n \,;\, |\xi - y| > \kappa)),$$

with $\kappa = \inf(1/i \,;\, i \in N)$.
Because $(\xi_t)$ is a diffusion and $\kappa > 0$, the right hand side is of order $\tau$ (Gihman and Skorohod, 1972, p. 64). As this situation may occur a countable number of times, we have to divide by $\kappa$, yielding order $(\tau/\kappa)$, of which the limit, $\tau \downarrow 0$, is zero. Q.E.D.

4.3 Proposition
Given the assumptions of theorem 4.2 are satisfied. Then for all $\xi_0 \in O'$, $(\xi_t)$ is a semimartingale strong Markov process, and its extended generator, $\mathcal{A}$, is

given by:

$$\mathcal{A}f = \mathcal{L}f + \mathcal{J}^+f \quad , \quad \text{for all } f\in\mathcal{D}(\mathcal{A}),$$

where $\mathcal{L}$ and $\mathcal{J}^+$ are given in proposition 2.3, while the domain of $\mathcal{A}$ is:

$$\mathcal{D}(\mathcal{A}) = (f \in C^{2,b}(0)\cap C^b(0\cup\partial 0);\ \mathcal{J}^+f(\xi)=0, \text{ all } \xi\in\partial 0).$$

Proof: Similar to the proof of proposition 3.3, except that now $\mathcal{J}^+f(\xi_s)=0$, for all $\xi_s\in 0-0'$, follows from $f\in C(0\cup\partial 0)$. Q.E.D.

Finally, we consider the more general situation with $\beta(\xi)\beta(\xi)^T$ being positive semidefinite. The construction of F works according to equations (17), (18), (20), (21) and (22), but with distance function:

$$d_\beta(\xi,\underline{\partial 0}) = \inf\ (r\geq 0;\ (\underline{\partial 0}\cap E_{\xi,r})\neq() \ ), \qquad (19)$$

where $\underline{\partial 0}$ is the subset of $\partial 0$ that is accessible by $(\xi',\tau)$ from 0, () is the empty set and $E_{\xi,r}$ is the closure of an n-dimensional ellipsoid, with centre $\xi+\alpha(\xi)\tau$ and shape defined by covariance $\beta(\xi)\beta(\xi)^T\tau$. Obviously, $d_\beta(.,\underline{\partial 0})$ is measurable, by which the $0_i$'s and 0' are Borel sets and F is measurable, and we get:

4.4 Corollary
Given an open subset 0 of $R^n$, and a mapping F, defined by (17) through (22). Then, under the assumptions of theorem 2.2, equation (IV) has for any $\xi_0\in 0'$ a pathwise unique solution $(\xi_t)$. Moreover, $(\xi_t)$ is then a Markov process, with sample paths being measurable on the stochastic basis $(\Omega,\mathcal{F},F,P)$.

Next, we come to the main result of this section.

4.5 Theorem
With probability one, the process $(\xi_t)$, of corollary 4.4, exits $0\cup\underline{\partial 0}$ zero times on $(0,\infty)$.

Proof:
By the definition of F, all points of p in $R^+$ become active as soon as $(\xi_t)$ has exit 0'. This situation holds on until $(\xi_t)$ reenters 0'. The reentering may occur due to drift and/or diffusion or due to a jump generated by a point of p in $R^+$. Obviously, the cases that $(\xi_t)$ reenters 0' by drift and/or diffusion without exit of $0\cup\underline{\partial 0}$ do not cause any difficulties. Of those cases where $\underline{\partial 0}$ is accessible through drift only, we follow the proof of theorem 3.1. Say $\underline{\partial 0_\alpha}$ is the subset of $\underline{\partial 0}$ that can only be entered by $(\xi',\tau)$ due to drift. For all other cases we then notice that a strictly positive type (19) distance $d_\beta$ at the moment of exit 0', corresponds with a strictly positive Euclidean distance from $\underline{\partial 0}-\underline{\partial 0_\alpha}$, due to the local boundedness of $|\alpha(\xi_t)|$ and $\|\beta(\xi_t)\|$. Subsequently, we may follow the proof of theorem 4.2 for these cases. Q.E.D.

4.6 Theorem
Given the assumptions of corollary 4.4 are satisfied. Then for all $\xi_0\in 0'$, $(\xi_t)$ is a semimartingale strong Markov process, and its extended generator, $\mathcal{A}$, is given by:

$$\mathcal{A}f = \mathcal{L}f + \mathcal{J}^+f \quad , \quad \text{for all } f\in\mathcal{D}(\mathcal{A}),$$

where $\mathcal{L}$ and $\mathcal{J}^+$ are those given in proposition 2.3, while the domain of $\mathcal{A}$ is:

$$\mathcal{D}(\mathcal{A}) = (f \in C^{2,b}(0)\cap C^b(0\cup\underline{\partial 0});\ \mathcal{J}^+f(\xi)=0 \text{ all } \xi\in\underline{\partial 0}).$$

Proof: Similar to the proofs of theorem 3.3 and proposition 4.3.

## 5. THE HYBRID STATE SPACE SITUATION

In this section we explicitly consider the hybrid state space situation for a system of the form (2.a,b), in such a way that there is no need of assuming a particular $\Psi$ or $\Lambda$. As such, all jump reflecting boundary results of the former sections fit into the results of this section. For ease of notation and interpretation, we rewrite the SDE form (2.a,b) by replacing the Poisson random measure, p, by a multivariate counting process, $\nu_t$, such that the pathwise uniqueness of (2)'s solution is preserved. We do that by defining, for all Borel $U\subset R^+xR^d$,

$$\nu_t(U) = \int_0^t\int_U \chi(\ [u_1<\Lambda(\xi_{s-})]\ \cup\ [F(\xi_{s-})\neq 0]\ )\ p(ds,du), \qquad (23.a)$$

and then rewriting (II) as

$$d\xi_t = \alpha(\xi_t)dt + \beta(\xi_t)dw_t + \int_{R^-xR^d}\Psi(\xi_{t-},u)\ q(dt,du) + \int_{R^+xR^d}\Psi(\xi_{t-},u)\ d\nu_t(du) \ . \qquad (23.b)$$

The main objective of this section is to show that the last term of (23.b) generates a particular type of jump: a jump in $(\xi_t)$ that anticipates a simultaneous switching of $(\xi^1_t)$. For short we refer to this type of jumps as hybrid jumps. Notice that these hybrid jumps are in some sense unexpected, as all coefficients of (23.a,b) are non-anticipating. To show these hybrid jumps explicitly, we need some preparation.

5.1 Lemma
Under assumptions A.1, A.2, A".3, A".4 and A".5, the pair of equations (23.a,b) has for any $\xi_0\in 0'$ a pathwise unique solution $(\xi_t,\nu_t)$, where $\nu_t$ is a multivariate counting process on $R_+xR^+xR^d$ of a predictable intensity, $\lambda_t\equiv\lambda(\xi_{t-})$. Moreover both $(\xi_t,\nu_t)$ and $(\xi_t)$ are then semimartingale strong Markov processes, of which $(\xi_t)$ is indistinguishable from the one in theorem 2.2.

Proof:
It follows from theorem 2.2, that the system of equations (2.a,b) and (23.a) has, for any Borel U, a pathwise unique solution $(\xi_t,\nu_t(U))$. With this, system (2.a,b), (23.a) has a pathwise unique solution $(\xi_t,\nu_t)$. Obviously all potentially active points of p, that are in $R_+xR^+xR^d$, are collected by $\nu_t$ in a predictable way, by which we can write

$$\int_{R^+xR^d}\Psi(\xi_{t-},u)\ \chi(\ [u_1<\Lambda(\xi_{t-})]\ \cup\ [F(\xi_{t-})\neq 0]\ ).$$

$$.p(dt,du) = \int_{R^+xR^d}\Psi(\xi_{t-},u)\ d\nu_t(du) \quad ,$$

up to indistinguishability. This implies that the solution of (2.b) is indistinguishable from the solution of (23.b). Q.E.D.

Now we are prepared to consider the hybrid state space situation. Therefore we assume that the first component of $\xi_t$ is M-valued, with $M\subset N=\{1,2,..\}$, and that we can write the first scalar equation of (23.b) as follows:

$$d\xi^1_t = \int_{R^+xR^d}\Psi_1(\xi_{t-},u)\ d\nu_t(du), \qquad (23.c)$$

with $\Psi_1$ a mapping of $R^nxR^+xR^d$ into the integer lattice, z.
Next we assume that $\Psi$ satisfies, for all $u_1\in(0,\Lambda(\xi))$,

$$\Psi(\xi,u) = \sum_{\eta\in M}\chi\left[\ \sum_{i=0}^{\eta-1}\lambda(i,\xi) \leq u_1 < \sum_{i=0}^{\eta}\lambda(i,\xi)\right]\varphi(\eta,\xi,\underline{u}), \qquad (24)$$

where $\varphi$ is a measurable mapping of $MxR^nxR^d$ into $zxR^{n-1}$, and $\lambda$ is a measurable mapping of $NxR^n$ into $R_+$, such that $\lambda(i,.)=0$ for all $i\in N-M$, and $\sum_{i\in N}\lambda(i,\xi)=\Lambda(\xi)$.
Moreover, we assume that for all $\eta\in N$, $\xi\in zxR^{n-1}$ and $\underline{u}\in R^d$,

$$\varphi_1(\eta,\xi,\underline{u}) = \eta-\xi_1 \ , \qquad (25)$$

which, together with (24) and $\lambda(i,.)=0$ for all $i\in N-M$, implies that if $\xi^1_0\in M$, then $(\xi^1_t)$ in $R_+xM$.
Substitution of (24) and (25) in (23.b,c) and subsequent evaluation yield

$$d\xi^1_t = \int_{R^+}\sum_{\eta\in M}\chi\left[\ \sum_{i=0}^{\eta-1}\lambda(i,\xi_{t-}) \leq u^*_{t-} < \sum_{i=0}^{\eta}\lambda(i,\xi_{t-})\right].$$

$$.(\eta-\xi^1_{t-})\ d\nu_t(du_1xR^d), \qquad (26.a)$$

with: $u^*_s \equiv u_1-k\Lambda(\xi_s)$,
for some integer k such that $0 < u^*_s \leq \Lambda(\xi_s)$,

$$d\underline{\xi}_t = \underline{\alpha}(\xi_t)dt + \underline{\beta}(\xi_t)dw_t + \int_{R^-xR^d}\underline{\Psi}(\xi_{t-},u)\ q(dt,du) + \int_{R^+xR^d}\underline{\varphi}(\xi^1_t,\xi_{t-},\underline{u})\ d\nu_t(R^+xd\underline{u}), \qquad (26.b)$$

$$\nu_t(U) = \int_0^t\int_U \chi(\ [u_1\leq\Lambda(\xi_{s-})]U[F(\xi_{s-})\neq 0]\ )\ p(ds,du), \qquad (26.c)$$

all Borel $U\subset R^+xR^d$, where underlining of a vector refers to all, but the first, components of that vector.

Assumptions

A.i Given, for all $\xi\in R^n-0'$ and $u\in R^+xR^d$,

$$\Lambda(\xi) = \sum_{\eta\in N}\lambda(\eta,\xi) = 1,$$

$$\varphi_1(\eta,\xi,\underline{u}) = \eta-\xi_1,$$
$$m(du) = du_1x\mu(d\underline{u}).$$

For all $k\in N$ there exists a constant $M_k$, such that for all $\xi\in B_k$.

$$\sum_{\eta\in N}\lambda(\eta,\xi)\ [|\eta-\xi_1| + \int_{R^d}|\underline{\varphi}(\eta,\xi,\underline{u})|\ \mu(d\underline{u})] \leq M_k.$$

A.5

a. For all $\xi \in O'$,
$\lambda(i,\xi)$ is continuous in $\xi$,
$\lambda(i,\xi)=0$ for all $i \in N-M$,
$\Lambda(\xi) \equiv \sum_{i \in N} \lambda(i,\xi)$ is uniformly bounded.

b. $(\xi_t)$, $t \in R_+$, exits $O'$ at most a countable number of times.

5.2 Theorem

Given the hybrid space $O' \equiv O' \cap (M \times R^{n-1})$.
Under assumptions A.1 through A.5, the system of equations (26.a,b,c) has for any $\xi_0 \in O'$ a pathwise unique solution $(\xi_t, \nu_t)$. Moreover $(\xi_t)$ is then a semimartingale strong Markov process in $R_+ \times O'$.

Proof:

Due to A.1 and A.5.a, (24) defines $\Psi$ as a measurable mapping (see proof of theorem 2.2), by which (26.a,b,c) is a special case of (23.a,b,c). Next we show that A.4 implies A".4, by which lemma 5.1 and (24) imply that the solution of (23.a,b,c) is indistinguishable from the solution of (26.a,b,c). To arrive at A".4, we start from A.4 and subsequently use A.5.a, interchange order of integration and substitute (24). Q.E.D.

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Due to its extensive form, equation (26.a,b,c) hides the results for which the above analysis has been carried out. Therefore, we take a closer look at it in case that p has no points in $R^-$. Then, (26.b) becomes

$$d\xi_t = g(\xi_t)dt + \beta(\xi_t)dw_t + \int_R d \varphi(\xi^1_t, \xi^1_{t-}, u) d\nu_t(R^+ x du) \qquad (27.a)$$

Moreover, to avoid the use of equations (26.a,c), we go over to the common descriptive way of formulating $(\nu_t)$ and $(\xi^1_t)$:
$(\nu_t)$ is a multivariate counting process characterized by the $\mathcal{F}_t$-predictable intensity, $\Gamma_t$,

$$\Gamma_t = \Lambda(\xi_{t-}) \left[1 + F(\xi_{t-}) \lim_{\tau \downarrow 0} 1/\tau\right], \qquad (27.b)$$

and a deterministic jump measure $\mu(du)$.
$(\xi^1_t)$ is a process with a countable state space, N, and with an $\mathcal{F}_t$-predictable rate, $r_{ij,t}$, of jumping from $\xi^1_{t-}=j$ to $\xi^1_t=i$, $i \neq j$,

$$r_{ij,t} \equiv \lambda(i,(j,\xi_{t-})) \left[1 + F(.,\xi_{t-}) \lim_{\tau \downarrow 0} 1/\tau\right], \qquad (27.c)$$

while $\sum_{i \neq j} r_{ij,t} \leq \Gamma_t$.

From this formulation, we easily notice the interesting effect that $\xi^1_t$ appears in the coefficient, $\varphi$, of (27.a)'s third right hand term. This means that $\varphi(\xi^1_t, \xi^1_{t-}, u)$ anticipates a switching from $\xi^1_{t-}$ to $\xi^1_t$, and thus a jump of $(\xi_t)$ anticipates a simultaneous transition of $(\xi^1_t)$. Verify that the anticipating coefficient $\varphi$ already appears in (26.a,b,c), while there is no anticipating coefficient in equation (23.a,b,c). As the solutions of both equations are indistinguishable, we conclude that (23.a,b,c) is the canonical representation of a system with hybrid jumps, while (26.a,b,c), with the anticipating coefficient, is the representation that is more useful when it comes to the realization of Markov models with hybrid jumps.

Remark: If $\lambda(.,(\xi_1,\xi))$ is $\xi$-invariant, then $(\xi^1_t)$ is a countable state Markov process. In this case (27.a) can straightforwardly be obtained from a classical system like (1) of which all coefficients are continuous. For the situation that $(\xi_t)$ is continuous, i.e. $\varphi=0$, see Brockett and Blankenship (1977). For some applications with hybrid jumps, i.e. $\varphi \neq 0$, see Sworder (1972), Blom (1984) and Mariton (1987).

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